Journal of Computational Physics 233 (2013) 480–508

Contents lists available at SciVerse ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

A coarse-grid projection method for accelerating incompressible flow computations

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ARTICLE INFO

Article history:
Received 30 July 2011
Received in revised form 13 April 2012
Accepted 6 September 2012
Available online 25 September 2012

Keywords:
Incompressible flows
Navier–Stokes equations
Coarse-grid projection
Fast Poisson solvers
Multigrid methods

ABSTRACT

We present a coarse-grid projection (CGP) method for accelerating incompressible flow computations, which is applicable to methods involving Poisson equations as incompressibility constraints. The CGP methodology is a modular approach that facilitates data transfer with simple interpolations and uses black-box solvers for the Poisson and advection–diffusion equations in the flow solver. After solving the Poisson equation on a coarsened grid, an interpolation scheme is used to obtain the fine data for subsequent time stepping on the full grid. A particular version of the method is applied here to the vorticity-stream function, primitive variable, and vorticity-velocity formulations of incompressible Navier–Stokes equations. We compute several benchmark flow problems on two-dimensional Cartesian and non-Cartesian grids, as well as a three-dimensional flow problem. The method is found to accelerate these computations while retaining a level of accuracy close to that of the fine resolution field, which is significantly better than the accuracy obtained for a similar computation performed solely using a coarse grid. A linear acceleration rate is obtained for all the cases we consider due to the linear-cost elliptic Poisson solver used, with reduction factors in computational time between 2 and 42. The computational savings are larger when a suboptimal Poisson solver is used. We also find that the computational savings increase with increasing distortion ratio on non-Cartesian grids, making the CGP method a useful tool for accelerating generalized curvilinear incompressible flow solvers.

1. Introduction

Computational studies of incompressible flow problems are important in basic scientific research, and for a multitude of engineering applications. In the decades since the first incompressible flow computations were performed, many successful algorithms have been proposed for computing these flows [1–6]. Incompressible flows are fluid flows in which the flow speed is small compared to the speed of sound. Interestingly, this simple condition introduces some computational difficulties. One basic approach to solving incompressible flow problems is to retain the full compressible form of the Navier–Stokes and continuity equations (e.g., artificial compressibility methods [7–12]). The other basic approach is to use pressure-based methods in which the Navier–Stokes equations are simplified by treating the fluid density as constant. In this approach a different difficulty arises: the continuity equation no longer involves the time derivative of the density, which was previously used to calculate the pressure field from an equation of state, so there is no longer an independent equation for the pressure [13]. The solution, then, is to construct the pressure field to guarantee that the continuity equation is satisfied (e.g., the vorticity-stream function formulation [14,15]), the marker and cell method [16], or projection algorithms for the primitive

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http://dx.doi.org/10.1016/j.jcp.2012.09.005

variable formulation of the problem [17–19]. This is accomplished by solving a Poisson equation to find the pressure field [17–30].

This latter approach has been given much attention in the literature [13], and is the approach we focus on here. The computational cost per time step of the pressure-based methods is that of solving a vector-valued parabolic-type advection–diffusion equation and a scalar-valued elliptic-type Poisson equation. The number of Poisson equations that must be solved at each time step varies with the method and the problem dimensions, but for all the pressure-based methods, solving the Poisson equation takes considerably more computational resources than solving the advection–diffusion time dependent equations, especially for large scale problems and high Reynolds number flows [29].

One straightforward way to accelerate incompressible flow simulations is to reduce the number of grid points for the most time consuming part of the problem, the elliptic solver. The coarse-grid projection (CGP) framework was first proposed by Lentine et al. [31] and successfully applied to three-dimensional flow simulations for computer games using finite volume methods on unstructured grids. Here, we apply the CGP method to the primitive variable fractional step, the vorticity-stream function, and the vorticity-velocity formulations for incompressible flow problems using finite differences on structured grids to analyze the accuracy and efficiency of the CGP framework for several benchmark flows in two and three dimensions. We also solve flow problems on distorted grids, and on generalized curvilinear grids. The cost of the flow computations is reduced by coarsening the resolution of the numerical grid on which the Poisson equation is solved by factors of two in each direction according to $M = 2^{-\ell}N$, where N is the fine resolution of the numerical grid on which the advection–diffusion part of the problem is solved, and M is the coarse resolution for the solution of the Poisson equation. When $\ell = 0$ no coarsening is applied and the CGP method reduces to the underlying incompressible flow solver method. For just one level of coarsening, in which $\ell = 1$, and $M = N/2$, we have found that, remarkably, there is no significant resulting loss of accuracy in the fine resolution field data. For each subsequent level of coarsening investigated ($\ell = 2$ and $\ell = 3$) there is a further gain in computational speed-up, and an associated reduction in the accuracy of the fine resolution field data. However, the results obtained by the CGP method are more accurate than those of the standard coarse simulations for all the cases. In the current study, the third-order Runge–Kutta [32,33] and the second-order central difference schemes are used for temporal and spatial discretizations, respectively. The method is general in nature and can be applied to any Poisson equation-based incompressible Navier–Stokes solver. The results demonstrate that it is possible to obtain high-accuracy fine resolution data at the price of a mostly coarse computation. The method can easily be applied to high Reynolds number turbulence simulations in three dimensions, where we expect the saving in computational time to be significant.

One of the important aspects of the proposed method is its flexibility and its ease of use with existing incompressible flow solvers. The method is independent of the choice of Poisson solver and the choice of solver used for the advection–diffusion part of the governing equations. Linear- or quadratic-rate acceleration can be obtained, depending on the choice of Poisson solver. In this study we use two types of efficient linear-cost Poisson solvers, the V-cycle multigrid iterative solver and the fast Fourier transform (FFT) based direct solver, as our black-box Poisson solvers. The computational savings reported in this study are greater for flow solvers that use suboptimal, quadratic-cost Poisson solvers, as is demonstrated in one of the benchmark problem in this study.

This paper is organized as follows: the mathematical models including the primitive variable formulation, the vorticity-stream function formulation, and the vorticity-velocity formulation of Navier–Stokes equations are presented in Section 2. In Section 3, the CGP method is developed and the joint flow solver algorithms are presented. In Section 4, the CGP flow solver algorithms are applied to several different two-dimensional benchmark flow problems: the Taylor–Green decaying vortex problem, the evaluation of a double shear layer, the merging of a pair of co-rotating vortices, two-dimensional decaying turbulence, the Taylor–Green vortex problem on a distorted grid, and laminar flow over a circular cylinder. This section also includes the application of the CGP method to three-dimensional Taylor–Green vortex flows, in which vortex stretching and tilting occurs. The results are compared to results obtained by performing the calculations using the basic flows solvers alone to test the validity of the CGP framework. Speed-ups in computational time ranging from 2 to 42 are found. Final conclusions and some comments about the effectiveness and applicability of the CGP method are presented in Section 5.

2. Governing equations

2.1. Primitive variable (PV) formulation

The primitive variable formulation of the governing equations for incompressible viscous flows in dimensionless form with index notation is:

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j \partial x_j} \quad (1)$$

$$\frac{\partial u_j}{\partial x_j} = 0 \quad (2)$$

where Re is the Reynolds number, u_i is the velocity component in the ith direction, and p is the pressure. We use the fractional step procedure [17–19], in which the first (predictor) step is to solve the advection–diffusion equation to obtain an