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hp-Multigrid as Smoother algorithm for higher order discontinuous Galerkin discretizations of advection dominated flows: Part I. Multilevel analysis

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ABSTRACT

The *hp*-Multigrid as Smoother algorithm (*hp-MGS*) for the solution of higher order accurate space–time discontinuous Galerkin discretizations of advection dominated flows is presented. This algorithm combines *p*-multigrid with *h*-multigrid at all *p*-levels, where the *h*-multigrid acts as smoother in the *p*-multigrid. The performance of the *hp-MGS* algorithm is further improved using semi-coarsening in combination with a new semi-implicit Runge–Kutta method as smoother. A detailed multilevel analysis of the *hp-MGS* algorithm is presented to obtain more insight into the theoretical performance of the algorithm. As model problem a fourth order accurate space–time discontinuous Galerkin discretization of the advection–diffusion equation is considered. The multilevel analysis shows that the *hp-MGS* algorithm has excellent convergence rates, both for steady state and time-dependent problems, and low and high cell Reynolds numbers, including highly stretched meshes.

1. Introduction

Discontinuous Galerkin finite element methods are well suited to obtain higher order accurate discretizations on unstructured meshes. The use of basis functions which are only weakly coupled to neighboring elements results in a local discretization which allows, in combination with *hp*-mesh adaptation, the efficient capturing of detailed structures in the solution, and is also beneficial for parallel computing. During the past decade this has stimulated a large amount of research in both the development and analysis of DG methods and resulted in a wide variety of applications. For an overview of various aspects of DG methods, see e.g. [6,12].

Space–time discontinuous Galerkin methods are a special class of DG methods in which space and time are simultaneously discretized using basis functions which are discontinuous, both in space and time. The resulting discretization belongs to the class of arbitrary Lagrangian Eulerian (ALE) methods, is implicit in time and fully conservative on moving and deforming meshes as occur in fluid–structure interaction and free boundary problems, see e.g. [14,29,30,33].

For higher order accurate DG discretizations the efficient solution of the algebraic system resulting from an implicit time discretization is, however, non-trivial, in particular for steady state solutions of advection dominated flows. For these problems standard iterative techniques, such as multigrid and Krylov subspace methods, are generally suboptimal, especially on highly stretched meshes in boundary layers. This lack of computational efficiency currently seriously hampers the

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application of higher order accurate DG methods to large scale industrial applications. An important reason for this relatively slow convergence rate is that the algebraic system resulting from a higher order accurate DG discretization has quite different mathematical properties compared to lower order discretizations. The straightforward application of iterative techniques originally developed for lower order methods is therefore generally not optimal and should be supported by a more detailed mathematical analysis.

The need for improved convergence rates in the iterative solution of higher order accurate DG discretizations has motivated the research presented in this and the companion article [32], to which we will refer as Part II. The objectives of this research are to develop, analyze and optimize new multigrid algorithms for higher order accurate space–time DG discretizations of advection dominated flows. For this purpose we introduce the *hp*-Multigrid as Smoother algorithm. This algorithm combines *p*-multigrid with *h*-multigrid at all *p*-levels, where the *h*-multigrid acts as smoother in the *p*-multigrid. The theoretical tool to investigate the performance of the *hp-MGS* algorithm will be a detailed multilevel analysis, which is the main topic of this article. In Part II we will use this analysis to optimize the semi-implicit Runge–Kutta smoother in the *hp-MGS* algorithm in order to account for the special features of higher order accurate DG discretizations. In addition, numerical simulations will be presented which show the excellent performance of the *hp-MGS* algorithm on a number of test cases, including thin boundary layers and non-constant coefficients. In this article we will focus on space–time DG discretizations, but the results and techniques can be straightforwardly extended to other types of implicit DG discretizations, both for steady state and time-accurate problems.

As background information we start with a brief overview of the main algorithms developed during the past decade for the iterative solution of higher order accurate DG discretizations of the compressible Euler and Navier–Stokes equations, which are important models for advection dominated flows. The main techniques to solve these equations have been multigrid and preconditioned Krylov methods, in particular flexible GMRES. In this article we will focus on multigrid methods. For preconditioned Krylov methods we refer to [7,23,26]. Multigrid methods can, however, also be efficient preconditioners for flexible GMRES, see e.g. [26].

Multigrid methods applied to higher order accurate DG discretizations can be classified as *p*-, *h*-, and *hp*-multigrid methods. In *p*-multigrid the coarser levels are obtained using a sequence of lower order discretizations, whereas in *h*-multigrid coarser meshes are used. Here *p* refers to the polynomial order of the basis functions in the DG discretization and *h* to the mesh size. Combinations of both methods result in *hp*-multigrid.

The main benefit of *p*-multigrid is its simplicity since at all levels the same mesh is used, which makes the implementation on unstructured meshes straightforward. Applications of *p*-multigrid to higher order accurate DG discretizations of advection dominated flows can be found in [2,8,17–19,21]. The resulting algebraic system at the coarsest *p*-multigrid level can, however, still be very large. For the Euler equations an implicit Euler time integration method at the $p = 0$ level, with GMRES in combination with an ILU preconditioner or an LU-SGS algorithm to solve the resulting algebraic system, is suitable [2,17]. For the compressible Navier–Stokes equations an *hp*-multigrid method is a better alternative [21,26]. In most studies of the compressible Navier–Stokes equations a polynomial order $p = 1$ is used at the coarsest level, which gives significantly better results than $p = 0$, see e.g. [26]. In this multigrid method the algebraic system at the coarsest *p*-level is solved with an *h*-multigrid method. For nonlinear problems it was concluded in [26] that the linear or Newton *h*-multigrid method is significantly more efficient as a coarse grid solver in *hp*-multigrid than the nonlinear Full Approximation Scheme.

A crucial element in both *p*- and *hp*-multigrid algorithms are the smoothers. Many different types of smoothers have been tested for higher order accurate DG discretizations. A serious problem with many of these smoothers is their lack of robustness. Often significant under-relaxation is necessary to ensure stability of the iterative method. Under-relaxation is, however, not necessary when block Jacobi and (symmetric) block Gauss–Seidel methods are used [8,18,21,26]. For problems containing boundary layers line smoothers are generally necessary to deal with large aspect ratio meshes [8,26]. Explicit and (semi)-implicit time integration methods have also been used as smoothers [2,3,16,25]. In particular, Runge–Kutta methods can be developed into efficient multigrid smoothers when they are used as pseudo-time integrators, which was originally proposed in [13], see also [20]. Since time-accuracy is not important in pseudo-time significant freedom is available to optimize Runge–Kutta smoothers for good multigrid performance [16,25,29].

The theoretical analysis of multigrid algorithms for DG discretizations of advection dominated flows has been quite limited. Many of these studies considered the advection–diffusion equation or linearized versions of the compressible Euler equations. The main analysis tool to understand the performance of the multigrid algorithm has been single grid and two-level Fourier analysis [8,9,16,18,24,25,33]. For a more general discussion of these techniques we refer to [11,28,35,37].

Despite this extensive research currently available multigrid algorithms for higher order DG discretizations do not yet achieve optimal performance. In this article we present therefore a new approach, viz. the *hp*-Multigrid as Smoother (*hp-MGS*) algorithm. The *hp-MGS* algorithm is an extension of the Multigrid as Smoother algorithm, which was originally proposed in [22,34], to higher order accurate DG discretizations. The main focus in this article is on the multilevel analysis of the *hp-MGS* algorithm, which is crucial to understand and optimize its performance. In the multilevel analysis three *p*-levels and three uniformly and three semi-coarsened *h*-levels are used in order to obtain accurate estimates of the operator norms and spectral radius of the *hp-MGS* multigrid error transformation operator. In Part II this analysis will be used to optimize the coefficients in the semi-implicit Runge–Kutta smoother for a fourth order accurate space–time DG discretization of the two-dimensional advection–diffusion equation.

The outline of this article is as follows. In Section 2 we briefly discuss the space–time DG discretization and in Section 3 we introduce the *hp-MGS* algorithm and the semi-implicit Runge–Kutta smoother. The multigrid error transformation