Journal of Computational Physics 231 (2012) 7695–7714

Contents lists available at SciVerse ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Sparse spectral-tau method for the three-dimensional helically reduced wave equation on two-center domains

Stephen R. Lau [a,*], Richard H. Price [b]

[a] Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131, United States
[b] Center for Gravitational Wave Astronomy and Center for Advanced Radio Astronomy, Department of Physics and Astronomy, University of Texas at Brownsville, Brownsville, TX 78520, United States

ARTICLE INFO

Article history:
Received 14 June 2011
Received in revised form 1 July 2012
Accepted 3 July 2012
Available online 14 July 2012

Keywords:
Spectral methods
Gravitational waves
Wave equation
Mixed-type problems
Preconditioning
Helical symmetry

ABSTRACT

We describe a multidomain spectral-tau method for solving the three-dimensional helically reduced wave equation on the type of two-center domain that arises when modeling compact binary objects in astrophysical applications. A global two-center domain may arise as the union of Cartesian blocks, cylindrical shells, and inner and outer spherical shells. For each such subdomain, our key objective is to realize certain (differential and multiplication) physical-space operators as matrices acting on the corresponding set of modal coefficients. We then achieve sparse realizations through the integration "preconditioning" of Coutsias, Hagstrom, Hesthaven, and Torres. Since ours is the first three-dimensional multidomain implementation of the technique, we focus on the issue of convergence for the global solver, here the alternating Schwarz method accelerated by GMRES. Our methods may prove relevant for numerical solution of other mixed-type or elliptic problems, and in particular for the generation of initial data in general relativity.

1. Introduction and preliminaries

1.1. Introduction

This paper describes spectral methods designed with a specific application in mind: numerical solution of a mixed-type problem arising in gravitational physics. In reviewing an ongoing program to construct helically symmetric solutions to the Einstein equations, this introduction recalls the origins of this problem below. However, this paper also serves another purpose; it demonstrates that spectral-tau *integration preconditioning*[1] yields accurate numerical solutions to the helically reduced wave equation (HRWE), a mixed-type, variable coefficient, linear partial differential equation (PDE) problem, here posed on a nontrivial three-dimensional (3D) domain. Ref. [1] offered spectral-tau integration preconditioning as a general-purpose strategy for spectral approximation of differential equations, and that reference provides the most thorough description and analysis of the technique; related techniques were explored in [2] (integration postconditioning) and [3] (nodal integration preconditioning), with applications described in, for example, Refs. [4,5]. Heretofore, spectral-tau integration preconditioning has primarily been studied either in the ODE context or for PDE problems posed on basic two-dimensional (2D) domains (rectangles, annuli, and disks); however, as a warmup to the 3D problem addressed here, we have earlier studied it in a multidomain context for the 2D HRWE [6]. While the 3D HRWE is of interest on its own, here it serves to demonstrate the general challenges

* Corresponding author.
E-mail addresses: lau@math.unm.edu (S.R. Lau), rprice@phys.utb.edu (R.H. Price).

[1] We use this term to refer to a specific technique reviewed below; however, insofar as our work is concerned the word *preconditioning* might be a misnomer. In any case, the technique does achieve *sparsification*, and this is the aspect of the technique we focus on here.

0021-9991/$ - see front matter
http://dx.doi.org/10.1016/j.jcp.2012.07.006

and conditioning issues encountered in applying spectral-tau integration preconditioning to a 3D multidomain problem. We now describe the physical problem motivating our work.

The advent of gravitational wave detection has driven theoretical studies of gravitational wave sources. A source that is possibly interesting for ground-based detectors, and perhaps the most exciting source for space-based detectors, is the inspiral of two comparable mass black holes and their merger to form a single black hole. The early stage of inspiral is modeled with reasonable accuracy by perturbations of the Newtonian analysis, and the post-merger stage can be analyzed with black hole perturbation theory. The most difficult stage to analyze is the intermediate stage, when a few orbits remain. This epoch of inspiral is too late for a modified Newtonian approach, but too early for black hole perturbation theory. Yet this is the epoch in which a large part of the gravitational wave energy is generated.

The importance of, and difficulty in, analyzing this epoch was the original motivation for an innovative approximation, the periodic standing wave (PSW) method. This approach uses the fully nonlinear field interactions, but models the binary compact objects as forever on circular orbits of constant radius. Therefore, both the compact source motions and the fields exhibit helical symmetry. Not only does this symmetry reduce the number of independent variables, it also changes the nature of the governing PDEs, turning the problem from the hyperbolic evolution of initial data to one of mixed-type that is elliptic near a rotation axis and hyperbolic well outside the axis and beyond the orbits in the wave zone of the system. More details of this astrophysical background are given in [6]. Here we only point out that recent supercomputer evolutions of initial black hole binary data have been run stably for many orbits in the intermediate epoch. See, for example, Refs. [7–23] (not an exhaustive list), and [24] for a recent review. Even the inspiral of binaries with large mass ratios [21,22] or high spins [23], both particularly challenging cases, can now be computed. To be sure, recent successes with purely hyperbolic numerical evolutions have undercut the original motivation for the PSW approximation. Nevertheless, there remains a niche for the PSW approximation for the following reasons. First, it should provide a test bench for understanding nonlinear gravitational radiation reaction as a local process. Second, a helically symmetric solution of the Einstein equations would be, of its own accord, of bewitching interest.

The numerical computation of PSW fields has, in fact, already been carried out, using a single grid and a unique method devised especially for the problem by one of us (RHP) and coworkers. These computations were done in a series of steps [25–29] moving from linearized scalar fields up to and including the nonlinear tensor fields of general relativistic gravity. However, the method used proved too limited in accuracy to be useful. Furthermore, despite the attractive simplicity of the computational method, its implementation for general relativistic tensor fields proved challenging. The astrophysical PSW problem, therefore, can be viewed as yet unsolved. The spectral methods described here are designed to solve this astrophysical problem to high accuracy. In any case, as mentioned above, our work is relevant as a successful use of spectral-tau integration preconditioning for the solution of PDEs (even of mixed-type) on nontrivial 3D domains. From this standpoint, the astrophysical problem simply provides a convenient application, with a particularly interesting feature. In the astrophysical problem, the region in which the PDEs are hyperbolic (the distant wave zone) is a region in which the PDEs have only very small nonlinearities. The strong linearities, near the source objects, are confined to a region in which the PDEs are elliptic. While we do not consider nonlinearities in this paper, the methods we introduce for our linear model problem deliver sufficient accuracy that nonlinearities can almost surely be included.

Multidomain spectral methods for the binary inspiral of compact relativistic objects are not new. In pioneering work, nodal (i.e. collocation) methods were used by Pfeiffer et al. [30,31] for the elliptical problem of constructing binary black hole initial data, and are now being used by the Caltech-Cornell-CITA collaboration (see, for example, [20]) in the hyperbolic evolution problem. This work, now highly developed, relies on SpEC (the Spectral Einstein Code [32]), a large C++ project chiefly developed by Pfeiffer, Kidder, and Scheel, but also involving many other researchers and developers. In fact, the domain decomposition of Refs. [30,31] has motivated our own choice. Nevertheless, to date the 3D problem we consider here has not been numerically solved via spectral methods.

Our previous study [6] applied a modal multidomain spectral-tau method to a model nonlinear problem of two strong field sources in binary motion with only two spatial dimensions. That study also relied on integration preconditioning, although the relevant linear systems were inverted by direct rather than iterative methods (which was possible since the 2D problem was less memory intensive). Our 2D study, a proof of concept, showed that high accuracy could be achieved with relatively modest memory and run-time requirements. Here we generalize our 2D method to 3D, that is to three spatial dimensions and one time dimension, reduced to a problem with three independent variables by the imposition of helical symmetry. Due to the larger set of modes needed for the 3D problem, iterative solution of the relevant linear system is now necessary. We use the generalized minimum residual method (GMRES) [33,34]. Since the amount of work and storage per iteration increases with the iteration count [33,34], preconditioning is a crucial issue (and here we mean further, one might even say *genuine*, preconditioning on top of the "integration preconditioning"). Through a multilevel preconditioning scheme, we will achieve near round-off accuracy for large truncations ($\simeq$ 600,000 unknowns) with modest iteration counts. Moreover, as we achieve a sparse formulation of the relevant linear system, each iteration is also fast.

1.2. Specification of the problem

Before writing down our mixed-type PDE problem, we describe the two-center (hereafter 2-center) domain $\mathcal{D}$ on which the problem is posed, first recalling the coordinate conventions of [26]. Let $\{x, y, z\}$ represent the inertial Cartesian system