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A staggered-grid convolutional differentiator for elastic wave modelling

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ABSTRACT

The computation of derivatives in governing partial differential equations is one of the most investigated subjects in the numerical simulation of physical wave propagation. An analytical staggered-grid convolutional differentiator (CD) for first-order velocity-stress elastic wave equations is derived in this paper by inverse Fourier transformation of the band-limited spectrum of a first derivative operator. A taper window function is used to truncate the infinite staggered-grid CD stencil. The truncated CD operator is almost as accurate as the analytical solution, and as efficient as the finite-difference (FD) method. The selection of window functions will influence the accuracy of the CD operator in wave simulation. We search for the optimal Gaussian windows for different order CDs by minimizing the spectral error of the derivative and comparing the windows with the normal Hanning window function for tapering the CD operators. It is found that the optimal Gaussian window appears to be similar to the Hanning window function for tapering the same CD operator. We investigate the accuracy of the windowed CD operator and the staggered-grid FD method with different orders. Compared to the conventional staggered-grid FD method, a short staggered-grid CD operator achieves an accuracy equivalent to that of a long FD operator, with lower computational costs. For example, an 8th order staggered-grid CD operator can achieve the same accuracy of a 16th order staggered-grid FD algorithm but with half of the computational resources and time required. Numerical examples from a homogeneous model and a crustal waveguide model are used to illustrate the superiority of the CD operators over the conventional staggered-grid FD operators for the simulation of wave propagations.

1. Introduction

Numerical simulation of elastic wave propagation in heterogeneous media plays an important role in the investigation of the Earth's subsurface structure. A number of numerical methods have been developed to simulate wave propagation, each with strengths and limitations.[1] One of the most popular algorithms is the finite-difference (FD) method (e.g. [1,30,67,68,42,29,2,65,60,52,53]) due to its simplicity and flexibility in numerical implementation and its ability to handle heterogeneous media. The Fourier (or Hartley) pseudo-spectral (PS) method, an accurate differentiation scheme using the

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1 Color versions of one or more of the figures in this paper are available online at http://dx.doi.org/10.1016/j.jcp.2015.08.017.

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fast Fourier transform (FFT), is another commonly used approach (e.g. [24,37,38,19,59,77,23,32,70]). However, the Fourier method requires much more computation time than the FD method (e.g. [18,15]).

Other numerical wave modelling methods include the Finite-element (FE) method (e.g., [48,36]), the spectral element (SE) method (e.g., [35,33,34,9]), and the boundary element (BE) method (e.g., [21,22,25]).

Each of these methods has its merits and drawbacks. These methods do not have restrictions on the material variability and can be very accurate when a sufficiently fine grid is used [7]. As mentioned above, the FD methods and the PS methods are relatively easy to be implemented. But these two methods are difficult to deal with rugged topography. In the FE and SE methods the implementations of the boundary condition are natural. They allow a more flexible treatment of heterogeneous media with complicated unstructured grids or elements, which is different from the regular or staggered grids of the FD and PS methods.

Staggered-grids, which define physical parameters on both regular and half grids, are widely used to improve the accuracy and stability in numerical seismic wave modelling [68,42,28,47]. Carcione [6] reported that the staggered-grid solution is noise-free in the dynamic range where regular grids generate artifacts for anisotropic and viscoelastic wave equation modelling. Moczo et al. [51] constructed an explicit, heterogeneous, 3D, fourth-order, displacement-stress FD scheme on a staggered-grid and investigated the stability and grid dispersion. Saenger et al. [61] and Saenger and Shapiro [62] derived another scheme called the rotated staggered-grid method, where all medium parameters were defined at appropriate positions within an elementary cell for the essential operations. Bohlen and Saenger [4] compared the accuracy of the staggered-grid method and the rotated staggered-grid FD method for modelling Rayleigh waves.

Liu and Sen [45,46] derived explicit and implicit staggered-grid finite-difference (FD) formulas for derivatives of first order with any order of accuracy by a plane wave theory and Taylor's series expansion. Using staggered temporal and spatial grids, Fang et al. [17] derived a wave-extrapolation operator using a low rank decomposition for a first-order system of wave equations and designed the corresponding FD scheme. Witte [69], Chen [10] and Zhao et al. [73] introduced the staggered-grid scheme with the PS method for seismic wave modelling. Fornberg [20] and Corrêa et al. [14] discussed the high-order FD and PS methods for arbitrary-order derivatives on staggered-grids. They reported that the staggered-grid odd-order derivatives will improve the accuracy of seismic wave modelling due to their improved localized nature and compactness resulting from the fast convergence of the FD operators.

In addition to the popular FD and PS methods, there is another method called the convolutional differentiator (CD) method, which combines the advantages of high accuracy of the PS method and computational efficiency of the FD method with relatively short operators. The CD method was introduced as a finite-impulse-response (FIR) design in the early 1970s (e.g. [56,49]). Its application in elastic seismic wave modelling was reported first by Mora [54]. An optimum design of spatial CD operators for wave equation computations was investigated by Holberg [27]. Fornberg [20] derived a general CD expression for any order derivatives for both centred- and staggered-grids through discrete spectral analysis. By means of the inverse Fourier transform, Zhou and Greenhalgh [74] and Zhou et al. [75] derived an explicit continuous CD of second derivatives for the scalar wave equation and the acoustic wave equation, respectively. Based on the elastic displacement wave equation, Zhou et al. [76] extended the work to elastic wave simulation by using the staggered-grid CDs of the first derivative. Li et al. [44] presented a convolutional generalized orthogonal polynomial differentiator for accurate and efficient modelling of seismic scalar wavefields. Li et al. [43] introduced the symplectic discrete singular convolution differentiator, which is a structure-preserving method of modelling of elastic waves and concentrates on temporal discretization, rather than spatial discretization.

In this paper, we first review the two common elastic wave equation systems, displacement equations and velocity-stress equations [67,68,42]. Then we derive a continuous first-derivative CD formula of any even order for centred- and staggered-grids by the inverse Fourier transformation of the band-limited spectrum of the first derivative operator. A window function is required to reduce the Gibbs effect resulting from truncating the infinite CD operators for elastic wave modelling. Next, we directly search for the optimal Gaussian window functions by minimizing the spectral error of the first derivative. Finally, we apply the derived CD to solve the first-order velocity-stress elastic wave equations on a staggered-grid system. Numerical modelling results for a homogeneous model and a crustal waveguide model are presented to demonstrate the advantages of the CD method over the FD approach.

2. 2D elastic wave equations

The second order elastic wave equations that describe the motion of P-waves and vertically polarized (SV) waves in 2D are given as (see Kelly et al. [30])

$$\begin{cases} \rho \dfrac{\partial^2 u_x}{\partial^2 t} = \dfrac{\partial}{\partial x}\left[(\lambda+2\mu)\dfrac{\partial u_x}{\partial x} + \lambda \dfrac{\partial u_z}{\partial z}\right] + \dfrac{\partial}{\partial z}\left[\mu\left(\dfrac{\partial u_x}{\partial z} + \dfrac{\partial u_z}{\partial x}\right)\right] \\ \rho \dfrac{\partial^2 u_z}{\partial^2 t} = \dfrac{\partial}{\partial z}\left[(\lambda+2\mu)\dfrac{\partial u_z}{\partial z} + \lambda \dfrac{\partial u_x}{\partial x}\right] + \dfrac{\partial}{\partial x}\left[\mu\left(\dfrac{\partial u_x}{\partial z} + \dfrac{\partial u_z}{\partial x}\right)\right] \end{cases} \tag{1}$$