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Multiwavelet-based grid adaptation with discontinuous Galerkin schemes for shallow water equations

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ABSTRACT

We provide an adaptive strategy for solving shallow water equations with dynamic grid adaptation including a sparse representation of the bottom topography. A challenge in computing approximate solutions to the shallow water equations including wetting and drying is to achieve the positivity of the water height and the well-balancing of the approximate solution. A key property of our adaptive strategy is that it guarantees that these properties are preserved during the refinement and coarsening steps in the adaptation process.

The underlying idea of our adaptive strategy is to perform a multiresolution analysis using multiwavelets on a hierarchy of nested grids. This provides difference information between successive refinement levels that may become negligibly small in regions where the solution is locally smooth. Applying hard thresholding the data are highly compressed and local grid adaptation is triggered by the remaining significant coefficients. Furthermore we use the multiresolution analysis of the underlying data as an additional indicator of whether the limiter has to be applied on a cell or not. By this the number of cells where the limiter is applied is reduced without spoiling the accuracy of the solution.

By means of well-known 1D and 2D benchmark problems, we verify that multiwavelet-based grid adaptation can significantly reduce the computational cost by sparsening the computational grids, while retaining accuracy and keeping well-balancing and positivity.

1. Introduction

The shallow water equations are a hyperbolic system of conservation laws which have been commonly used to model shallow free surface water flows. These are given by

$$\begin{pmatrix} h \\ h\mathbf{v} \end{pmatrix}_t + \nabla \cdot \begin{pmatrix} h\mathbf{v} \\ h\mathbf{v} \otimes \mathbf{v} + \frac{1}{2} g h^2 I \end{pmatrix} = \begin{pmatrix} 0 \\ -gh\nabla b \end{pmatrix}, \tag{1}$$

where h denotes the water height and $\mathbf{v}$ is the velocity vector. The bottom topography is represented by b, g is the gravitational acceleration and I is the $d \times d$ identity matrix, where d is the number of space dimensions.

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The shallow water equations have been widely discretized using finite volume (FV) schemes providing good results, see for instance [1]. First order FV schemes are widely found in the literature [2–8]. The conceptual simplicity of first order FV schemes and its rather straightforward implementation and ease for parallelization have made it very popular. However, the first order FV approach suffers from high sensitivity to the computational mesh [9], thus requiring high mesh resolution to ensure accuracy. Higher order FV methods have been attempted to solve this issue. In many cases they remain at second order [10–12]. When higher order is achieved [13–15] they introduce a new problem, the need to enlarge the computational stencil and thus the loss of local support. In particular, this requires more data transfer on distributed memory architectures and, thus, affects the efficiency of parallel implementations.

Discontinuous Galerkin (DG) schemes have been recently applied to the shallow water equations [16–23]. The DG method generalizes the concept of the FV method, while relying on the Finite Element notion of projecting the solution onto a space of trial functions, but, without the restriction of keeping the functions continuous. Therefore, the piecewise discontinuous description of the solution allows to achieve high order accuracy while keeping locality. Furthermore, the discontinuous interfaces at the cell edges require the solution of Riemann problems employing approximate Riemann solvers [24] and related techniques [25–27] that have been widely studied in the context of FV schemes and shallow water equations.

Important issues relating to shallow water equations must also be addressed. In particular, a DG scheme for the shallow water equations must be well-balanced [28–30] and positivity-preserving [31], i.e., it must be able (i) to keep steady states over non-constant topography, and (ii) to keep positive values of depth, specially near the wet/dry front. It is also of great importance to have a sufficiently high resolution representation of the topography at very large gradients [9]. This has led to the need of not only properly discretizing the bed source term, but to build meshes that can harbor the required information with precision.

The high order nature of DG methods allows for accurate solutions on coarser meshes than first order FV schemes. However, the computational overhead introduced by the required operations for DG, as well as the increased number of time steps required due to a lower CFL number can make DG very expensive even compared to a FV approximation on a fine mesh. Therefore, in order to improve efficiency, adaptive meshes can be designed that will refine on those regions in which the solution shows large gradients or even discontinuities and coarsen the mesh on those regions where the solution is smooth. In this way, high accuracy can be obtained while reducing computational effort. Furthermore, the high order approximation can also be used to represent the topography, thus also reducing the need of high resolution meshes.

In order to control local grid refinement numerous refinement indicators have been developed, for instance, they are based on interpolation error estimates of some key quantity using a priori knowledge of the solution. In [32,33] Remacle et al. present a second order scheme for shallow water equations with anisotropic grid adaptivity on triangular meshes, but without addressing well-balancing and positivity-preserving. Since these concepts offer no reliable error control, a priori as well as a posteriori error estimates have been developed to control the adaptive process, e.g., Bey and Oden [34], Adjerid et al. [35], Houston et al. [36–38], Dedner et al. [39], and recently Mavriplis et al. [40].

These approaches are aiming at estimating the error of the solution. Since there are in general no rigorous error estimates available for nonlinear systems of conservation laws, the concept of *multiresolution-based grid adaptation* has been developed in the context of FV schemes [41,42]. The idea of a multiresolution representation of data originates from the field of wavelets [43–46]. The starting point for a multiresolution analysis is a hierarchy of nested grids at different resolutions. Consider two different levels of resolution in this hierarchy, the cell averages on the finer level can be represented as a combination of the cell averages on the coarser level and of *detail coefficients* encoding the difference between the two levels. Since detail coefficients may become negligibly small in regions where the underlying data are locally smooth one can locally decide if the additional resolution of the finer mesh is needed. If these details are sufficiently small the solution can be represented on the coarser grid without significant loss of accuracy. Therefore the size of these details is a good indicator for adaptivity and limiting.

Multiscale-based mesh adaptivity has been quite successful with FV solvers for compressible fluid flow, cf. [47–50]. In recent years the concept was extended to the framework of DG schemes. Originally in [51,52] the concept was analytically and numerically investigated for the one-dimensional scalar case. The method was then extended to the Euler equations [53] and later on to the multi-dimensional case [54]. We track this idea and apply it to shallow water equations and add modifications in order to address common problems in solving shallow water equations, i.e., make sure that the presented grid adaptation is well-balanced and positivity-preserving. In [55] multiresolution-based mesh adaptation for shallow water flows in the context of FV schemes is discussed. Additionally we address the problem of wetting and drying. In [22,23,56] discontinuous Galerkin schemes are applied to shallow water equations on cubed spheres considering smooth problems and no bottom topography or wetting and drying. In [56] this is combined with multiwavelet analysis for filtering purposes. The evolution is performed on the multiscale coefficients instead of the single-scale coefficients. This approach is based on [57] where the strategy was applied to convection and convection–diffusion problems. For shallow water modeling, the mesh must adapt to both the solution and the topography, since the latter affects significantly the first. This is also treated in the method proposed herein. In addition, we use the multiresolution analysis of the data in the grid adaptation as an additional indicator for limiting to avoid limiting in areas, where the solution is smooth: The limiter is only applied to cells which are refined up to the highest refinement level. In [58] the multiresolution analysis is used as an indicator for limiting as well, but a different strategy is used to decide whether the solution on a cell is limited or not.

This paper is structured as follows. Section 2 provides a brief description of the DG method. Well-balancing and positivity-preserving is described in Section 3. In Section 4 we introduce the multiresolution analysis using multiwavelets