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Towards consistence and convergence of conservative SPH approximations

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ABSTRACT

Typical conservative smoothed particle hydrodynamics (SPH) approximations of the gradient of a scalar field introduce two errors: one (smoothing error) is due to smoothing of the gradient by an integration associated with a kernel function; the other (integration error) is due to approximating the integration by summation over all particles within the kernel support. When particles are not on a uniform grid, the integration error leads to violation of zero-order consistency, i.e. the inability to reproduce a constant field. In this paper we confirm that partition of unity is the condition under which the conservative SPH approximation achieves both consistence and convergence. We show that this condition can be achieved by relaxing a particle distribution under a constant pressure field and invariant particle volume. The resulting particle distribution is very similar to that is typical for liquid molecules. We further show that with two different typical kernel functions the SPH approximation, upon satisfying the partition of unity property, is able to achieve very high-order of the integration error, which previously could be shown only with particles on a uniform grid. The background pressure used in a weakly compressible SPH simulation implies a self-relaxation mechanism, which explains that convergence with respect to increasing particle numbers could be obtained in SPH simulations, although not predicted by previous numerical analysis. Furthermore, by relating the integration error to the background pressure, we explain why the previously proposed transport-velocity formulation of SPH (S. Adami et al. (2013) [1]) is able to achieve unprecedented accuracy and stability.

1. Introduction

Smoothed particle hydrodynamics (SPH) is a mesh-free Lagrangian simulation method introduced in Refs. [11] and [4]. It has a wide range of applications ranging from astrophysics to complex multi-phase flows [13,14]. In SPH the hydrodynamic equations are discretized by particles, and each particle generally can assume an arbitrary location in space. At a particle location $\mathbf{r}_a$, the SPH approximation of the gradient of a smooth scalar field $\psi(\mathbf{r})$ is obtained by the following three steps,

$$\nabla\psi(\mathbf{r}_a) \approx \int_V \nabla\psi(\mathbf{r}) W(\mathbf{r}_a - \mathbf{r}, h)\, dV = -\int_V \psi(\mathbf{r}) \nabla W(\mathbf{r}_a - \mathbf{r}, h) dV \approx -\sum_b \psi(\mathbf{r}_b) \nabla W(\mathbf{r}_a - \mathbf{r}_b, h) V_b, \tag{1}$$

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where $W(\mathbf{r}, h)$ is a kernel function with compact support with the general radially symmetric form

$$W(\mathbf{r}, h) = h^{-d} w(s), \tag{2}$$

where $s = r/h$ and $r = |\mathbf{r}|$, and d is the number of spatial dimensions. The parameter h is a smoothing length, and when it tends to zero $W(\mathbf{r}, h)$ approaches the Dirac delta function. The first stage in Eq. (1) is called filtering or smoothing approximation, and the error introduced by this stage is called smoothing error E_s, which is the difference between the smoothing approximation and the exact derivative. The second stage is integration by parts assuming that the kernel function vanishes at the support boundaries. The third stage approximates the integral by summation over all particles, where V_b denotes the volume of a particle b, within the kernel support. The error introduced by this stage is called integration error E_r, which is the difference between the final SPH approximation and the smoothing approximation. Therefore, the overall truncation error of the SPH approximation is $E_t = E_s + E_r$.

Monaghan [13] showed that for the B-spline and quintic kernels $E_s = O(h^2)$. Quinlan et al. [17] analyzed E_r analytically for particles on a uniform grid and numerically when these particles are randomly perturbed. For particles on a uniform grid, or ordered particles, $E_r = O[(\Delta/h)^{\beta+2}]$, where Δ is the grid size and β is determined by the smoothness (vanishing derivatives up to order β) of the kernel-support boundary. For randomly perturbed particles, Quinlan et al. [17] showed that the SPH approximation is not zero-order consistent, i.e. it does not reproduce the vanishing derivative of a constant-valued function. This inconsistency is introduced by E_r, and does not approach zero as Δ/h decreases. The failure of zero-order consistency is often considered as a major drawback of SPH [17,10]. Remedies have been proposed, such as the reproducing kernel (RK) method [10,18]. When applied for SPH, the RK method reproduces a constant pressure field, it violates, however, the momentum conservation property, one of the most important properties of the original SPH method [13,14], as particle–particle interactions are no longer antisymmetric. In practice, SPH simulations exhibit convergent behavior with increasing particle numbers as long as the total momentum is conserved [15,7,16].

In this paper, we address this apparent contradiction and consider a typical anti-symmetric formulation based on Eq. (1) for the conservative discretization of pressure forces in the hydrodynamic equations. We introduce inter-particle faces that separate the volume of a particle from its neighboring particles. Based on a simple integration rule we show that zero-order consistency and asymptotic decay of E_r can be achieved when volumes defined by the particles and the inter-particle faces partition the entire domain, i.e. constitute a partition of unity. We show that this condition, without compromising the conservation property, can be achieved by relaxing particles under a constant pressure field and with invariant particle volume. We also demonstrate by numerical experiments that the convergence rate of E_r for such a relaxed distribution is of the same high order as for particles on a uniform grid. By comparing the particle relaxation approach and the estimation of E_r, we explain why many SPH simulations in practice show clearly convergent behavior with increasing particle numbers, and why a recently developed SPH formulation [1] is able to achieve unprecedented numerical accuracy and stability.

2. Anti-symmetric formulation of the SPH approximation

A typical anti-symmetric formulation of the SPH approximation based on Eq. (1) is obtained by

$$\nabla\psi_a = \nabla\psi_a + \psi_b\nabla 1 \approx -\sum_b (\psi_a + \psi_b)\nabla W_{ab} V_b = -2\sum_b \overline{\psi}_{ab}\nabla W_{ab} V_b \tag{3}$$

where $\nabla\psi_a \equiv \nabla\psi(\mathbf{r}_a)$, $\psi_a \equiv \psi(\mathbf{r}_a)$, $\nabla W_{ab} \equiv \nabla W(\mathbf{r}_a - \mathbf{r}_b, h)$ and $\overline{\psi}_{ab} \equiv (\psi_a + \psi_b)/2$. Note the anti-symmetric property of the derivative of the kernel function, i.e. $\nabla W_{ab} = -\nabla W_{ba}$. With Eq. (3) the SPH discretization for computing the pressure forces acting on a particle assumes the form

$$\mathbf{F}_a = V_a \nabla p_a \approx -2\sum_b \overline{p}_{ab}\nabla W_{ab} V_a V_b, \tag{4}$$

which implies momentum conservation of the particle system by

$$\sum_a \mathbf{F}_a = -2\sum_a\sum_b \overline{p}_{ab}\nabla W_{ab} V_a V_b = 0. \tag{5}$$

If only one particle pair is considered, the interaction force is

$$\mathbf{F}_{ab} = -2\overline{p}_{ab}\nabla W_{ab} V_a V_b = -\overline{p}_{ab} A_{ab}\mathbf{e}_{ab}, \tag{6}$$

where $\mathbf{e}_{ab}$ is the unit vector from particle a to particle b. Since $\overline{p}_{ab}$ and $A_{ab}\mathbf{e}_{ab}$ can be considered as inter-particle pressure and inter-particle face area, respectively [6], Eq. (4) can be considered as an approximation of the surface integral over all the inter-particle faces between particle a and its neighbors within the support. The transformation between the volume integral of Eq. (3) to the surface integral of Eq. (4) is done by multiplying V_a, the volume of particle a. Note that, the volume of a particle is defined implicitly, i.e. although the magnitudes of the volume V_a and the inter-particle face area $A_{ab}\mathbf{e}_{ab}$ are known, non-trivial reconstruction is required to determine the exact form of inter-particle faces.