Journal of Computational Physics 301 (2015) 456–483

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

A discontinuous Petrov–Galerkin methodology for adaptive solutions to the incompressible Navier–Stokes equations

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ARTICLE INFO

Article history:
Received 20 October 2014
Received in revised form 6 July 2015
Accepted 7 July 2015
Available online 14 July 2015

Keywords:
Discontinuous Petrov Galerkin
Incompressible flow
Navier–Stokes equations
Adaptive finite elements

ABSTRACT

The discontinuous Petrov–Galerkin methodology with optimal test functions (DPG) of Demkowicz and Gopalakrishnan [18,20] guarantees the optimality of the solution in an energy norm, and provides several features facilitating adaptive schemes. Whereas Bubnov–Galerkin methods use identical trial and test spaces, Petrov–Galerkin methods allow these function spaces to differ. In DPG, test functions are computed on the fly and are chosen to realize the supremum in the inf–sup condition; the method is equivalent to a minimum residual method. For well-posed problems with sufficiently regular solutions, DPG can be shown to converge at optimal rates—the inf–sup constants governing the convergence are mesh-independent, and of the same order as those governing the continuous problem [48]. DPG also provides an accurate mechanism for measuring the error, and this can be used to drive adaptive mesh refinements.

We employ DPG to solve the steady incompressible Navier–Stokes equations in two dimensions, building on previous work on the Stokes equations, and focusing particularly on the usefulness of the approach for automatic adaptivity starting from a coarse mesh. We apply our approach to a manufactured solution due to Kovasznay as well as the lid-driven cavity flow, backward-facing step, and flow past a cylinder problems.

1. Introduction

1.1. Motivation

Typical solutions of incompressible flow problems involve both fine- and large-scale phenomena, so that a uniform finite element mesh of sufficient granularity will at best be wasteful of computational resources, and at worst be infeasible because of resource limitations. Thus adaptive mesh refinements are desirable. In industry, the adaptivity schemes used are often ad hoc, requiring a domain expert to predict features of the solution. A badly chosen mesh may cause the code to take considerably longer to converge, or fail to converge altogether. Typically, the Navier–Stokes solver will be just one component in an optimization loop, which means that any failure requiring human intervention is costly.

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http://dx.doi.org/10.1016/j.jcp.2015.07.014

Our aim, therefore, is to develop a solver for the incompressible Navier–Stokes equations that provides robust adaptivity starting from a coarse mesh.[1] By robust, we mean both that the solver always converges to a solution in predictable time, and that the adaptive scheme is independent of the problem—no special expertise is required for adaptivity. In each of the adaptive experiments in the present work, we begin with a mesh that simply captures the geometry, perform some preliminary refinements to ensure that elements are roughly isotropic, and proceed with the automatic adaptivity algorithm described below.

1.2. Approach

The cornerstone of our approach will be the discontinuous Petrov–Galerkin with optimal test functions (DPG) finite element methodology recently developed by Leszek Demkowicz and Jay Gopalakrishnan [18,20]. Whereas Bubnov–Galerkin methods use the same function space for both test and trial functions, Petrov–Galerkin methods allow the spaces for test and trial functions to differ. In DPG, the test functions are computed on the fly and are chosen to minimize the residual. For a very broad class of well-posed problems, DPG offers provably optimal convergence rates with a modest stability constant—the "inf–sup" constants governing the convergence are mesh-independent, and of the same order as those governing the continuous problem [48]. In some of our experiments, DPG not only achieves the optimal rates, but gets very close to the best solution available in the discrete space. DPG also provides an accurate mechanism for measuring the error, and this can be used to drive adaptive mesh refinements.

Previously, we have studied DPG applied to the Stokes problem in some detail, with theoretical results predicting optimal rates of convergence, and numerical results that appear to show even more: it appears that we asymptotically approach the best approximation error available in the discrete space [48]. We began with Stokes because the Stokes equations are more susceptible than Navier–Stokes to rigorous mathematical analysis; our strategy is to use the theory developed for Stokes to guide the practical application to Navier–Stokes. Because of the success with the Stokes equations and their close relationship to the incompressible Navier–Stokes equations, we are optimistic that DPG can achieve good results with the latter as well.

Central to our study of these problems has been the use and further development of *Camellia* [47], a toolbox we developed for solving DPG problems which uses Sandia's Trilinos library of packages [29]. At present, Camellia supports arbitrary 1D meshes, 2D meshes of triangles and quads, and 3D meshes of hexahedra, provides mechanisms for easy specification of DPG variational forms, supports h- and p-refinements, and employs a distributed stiffness matrix and solution representation, among other features. In the future, we plan to enhance support for meshes of arbitrary spatial dimension, and add support for space–time elements and a distributed mesh representation.

1.3. Literature review: adaptive Navier–Stokes

The application of adaptive mesh refinement to incompressible flow problems is far from new. Here, we mention a few relevant references for finite element methods, spectral element methods, and least squares methods. DPG is a Petrov–Galerkin finite-element method (FEM), a generalization of the classical (Bubnov) Galerkin method. As early as 1993, Oden presented an approach for *hp*-adaptive FEM for the incompressible Navier–Stokes equations [42].

Spectral element methods (SEM) employ basis functions with global support, in contrast to FEM, which employ basis functions with support limited to the element. Karniadakis and Sherwin have produced a compendium of *hp*-adaptive SEM for computational fluid dynamics [31]. Galerkin FEM and SEM for incompressible flow problems require careful selection of velocity and pressure spaces; unstable discretizations can result in locking or non-convergence. This contrasts with the present work, in which we use equal-order discretizations for velocity and pressure, and for smooth solutions we obtain optimal rates of convergence in both variables. Moreover, DPG allows the discretizations for velocity and pressure to be chosen independently.

Least-squares finite and spectral element methods (LSFEM and LSSEM) employ formulations that minimize a residual, typically in the L^2 norm. Like DPG, least-squares methods allow independent selection of velocity and pressure discretizations. Recently Ozcelikkale and Sert have applied *hp*-adaptive LSSEM to model problems in 2D incompressible flow [43]. Like least-squares, DPG is a minimum-residual method—in fact, DPG can be understood as a least-squares method which minimizes the DPG energy norm. In contrast to classical least-squares methods, DPG's energy norm can be prescribed by appropriate selection of the norm on the test space, allowing it to avoid some of the problems classically exhibited by least-squares, such as over-diffusivity.

[1] Ultimately, we would like to produce such a solver for incompressible Navier–Stokes in arbitrary dimensions, for a range of Reynolds numbers limited only by the numerical precision of the machine. The scope of the present work is more modest: we limit ourselves to steady two-dimensional flows; the largest Reynolds number that we employ is 10^4. We restrict ourselves thus both to maintain a focused discussion and because transient and three-dimensional flows impose additional implementation challenges. In upcoming work, we plan to address transient and three-dimensional problems.