Journal of Computational Physics 231 (2012) 2482–2497

Contents lists available at SciVerse ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Numerical methods for stochastic partial differential equations with multiple scales

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ARTICLE INFO

Article history:
Received 23 May 2011
Received in revised form 3 October 2011
Accepted 27 November 2011
Available online 13 December 2011

Keywords:
Stochastic partial differential equations
Multiscale methods
Averaging
Homogenization
Heterogeneous multiscale method (HMM)

ABSTRACT

A new method for solving numerically stochastic partial differential equations (SPDEs) with multiple scales is presented. The method combines a spectral method with the heterogeneous multiscale method (HMM) presented in [W. E, D. Liu, E. Vanden-Eijnden, Analysis of multiscale methods for stochastic differential equations, Commun. Pure Appl. Math., 58(11) (2005) 1544–1585]. The class of problems that we consider are SPDEs with quadratic nonlinearities that were studied in [D. Blömker, M. Hairer, G.A. Pavliotis, Multiscale analysis for stochastic partial differential equations with quadratic nonlinearities, Nonlinearity, 20(7) (2007) 1721–1744]. For such SPDEs an amplitude equation which describes the effective dynamics at long time scales can be rigorously derived for both advective and diffusive time scales. Our method, based on micro and macro solvers, allows to capture numerically the amplitude equation accurately at a cost independent of the small scales in the problem. Numerical experiments illustrate the behavior of the proposed method.

1. Introduction

Many interesting phenomena in the physical sciences and in applications are characterized by their high dimensionality and the presence of many different spatial and temporal scales. Standard examples include atmosphere and ocean sciences [1], molecular dynamics [2] and materials science [3]. The mathematical description of phenomena of this type quite often leads to infinite dimensional multiscale systems that are described by nonlinear evolution partial differential equations (PDEs) with multiple scales.

Often physical systems are also subject to noise. This noise might be either due to thermal fluctuations [4], noise in some control parameter [5], coarse-graining of a high-dimensional deterministic system with random initial conditions [6,7], or the stochastic parameterization of small scales [8]. High dimensional multiscale dynamical systems that are subject to noise can be modeled accurately using stochastic partial differential equations (SPDEs) with a multiscale structure. As examples of mathematical modeling using SPDEs we mention the stochastic Navier–Stokes equations [9] that arise in the study of hydrodynamic fluctuations, the stochastic Swift–Hohenberg and stochastic Kuramoto–Shivashinsky equation that arise in the study of pattern formation [10], the Langevin-type SPDEs that arise in path sampling and Markov Chain Monte Carlo in infinite dimensional dimensions [11] and the stochastic KPZ equation that is used in the modeling of the evolution of growing interfaces. Most of the equations mentioned above are semilinear parabolic equations with quadratic nonlinearities for which the numerical algorithm proposed in this paper can be applied, in principle.

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0021-9991/$ - see front matter
doi:10.1016/j.jcp.2011.11.039

There are very few instances where SPDEs with multiple scales can be treated analytically. The goal of this paper is to develop numerical methods for solving accurately and efficiently multiscale SPDEs. Several numerical methods for SPDEs have been developed and analyzed in recent years, e.g. [12–14], based on a finite difference scheme in both space and time. It is well known that explicit time discretization via standard methods (e.g., as the Euler–Maruyama method) leads to a time-step restriction due to the stiffness originating from the discretisation of the diffusion operator (e.g. the Courant–Friedrichs–Lewy (CFL) condition $\Delta t \leqslant C(\Delta x)^2$, where Δt and Δx are the time and space discretization, respectively). The situation is even worse for SPDEs with multiple scales (e.g. of the form (3) and (4) below) as in this case the CFL condition becomes $\Delta t \leqslant C(\Delta x \cdot \epsilon)^2$, where $\epsilon \ll 1$ is the parameter measuring scale separation. Standard explicit methods become useless for SPDEs with multiple scales.

Such time-step restriction can in theory be removed by using implicit methods as was shown in [14]. However the implicitness of the numerical scheme forces one to solve potentially large linear algebraic problems at each time step. Furthermore, it was shown in [15] that implicit methods are not suited for studying the long time dynamics of fast–slow stochastic systems as they do not capture the correct invariant measure of the system. Although this result has been obtained for finite dimensional stochastic systems, it is expected that it also applies to infinite dimensional fast–slow systems of stochastic differential equations (SDEs), rendering the applicability of implicit methods to SPDEs with multiple scales questionable. We also note that a new class of explicit methods, the S-ROCK methods, with much better stability properties than the Euler–Maruyama method was recently introduced in [16–18]. Although these methods are much more efficient than traditional explicit methods, computing time issues will occur when trying to solve SPDEs with multiple scales as considered here, since the stiffness is extremely severe for small ϵ. Furthermore, capturing the correct invariant measure of the SPDE for $\Delta t > \epsilon$ is still an issue for such solvers.

In this paper we consider SPDEs of the form

$$\partial_t v = \mathcal{A} v + F(v) + \epsilon Q \xi, \tag{1}$$

posed in a bounded domain of $\mathbb{R}$ with appropriate boundary conditions. The differential operator $\mathcal{A}$ is assumed to be a non-positive self-adjoint operator in a Hilbert space $\mathcal{H}$ with compact resolvent, ξ denotes space-time Gaussian white noise, Q is the covariance operator of the noise and we take $\epsilon \ll 1$. We assume that the operator $\mathcal{A}$ and the covariance operator of the noise Q commute, and that $\mathcal{A}$ has a finite dimensional kernel.[1]

The finite dimensional kernel of the operator $\mathcal{A}$ leads to scale separation between the slow dynamics in $\mathcal{N}$ and the fast dynamics in the orthogonal complement of the null space $\mathcal{N}^\perp$, where $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}^\perp$. In this paper we will furthermore assume that noise acts directly only on the orthogonal complement $\mathcal{N}^\perp$. This assumption is consistent with the scaling of the noise in (1), i.e. that it is of $O(\epsilon)$, and it leads to the singularly perturbed SPDEs (3) and (4) below. When noise acts also on the slow variables, then, its amplitude has to be scaled differently in Eq. (1); in particular it has to be of $O(\epsilon^2)$. In this case, and after rescaling in time, we end up with a non-singularly perturbed equation for which the analysis is easier. A problem of this type has been studied and the amplitude equation has been derived in [39].

For concreteness, we will focus on the class of SPDEs with quadratic nonlinearities that was considered in [19], and assume that

$$F(u) = f(u) + \epsilon^\alpha g(u), \tag{2}$$

where f is a quadratic function (e.g. $f(u) = B(u,u)$, a symmetric bilinear form), g a linear function and the exponent α is either 1 or 2.[2] The choice of α will depend on the particular scaling. In order to describe the longtime behavior of the SPDEs we perform an *advective* rescaling set $v(t) := \epsilon u(\epsilon t)$. Using the scaling properties of white noise and (2) with $\alpha = 1$ we obtain the following singularly perturbed SPDE

$$\partial_t u = \frac{1}{\epsilon} \mathcal{A} u + F(u) + \frac{1}{\sqrt{\epsilon}} Q \xi. \tag{3}$$

Another scaling is of interest, namely the *diffusive* rescaling $v(t) := \epsilon u(\epsilon^2 t)$ which, for (2) with $\alpha = 2$ leads to the SPDE

$$\partial_t u = \frac{1}{\epsilon^2} \mathcal{A} u + \frac{1}{\epsilon} F(u) + \frac{1}{\epsilon} Q \xi. \tag{4}$$

Alternatively, one can start with the singularly perturbed SPDEs (3) and (4) without any reference to the SPDE (1).

Singularly perturbed SPDEs with quadratic nonlinearities provide a natural testbed for testing the applicability of the heterogeneous multiscale method to infinite dimensional stochastic systems, since a rigorous homogenization theory exists for this class of SPDEs [19]. Furthermore, SPDEs of this form arise naturally in stochastic models for climate [1] and in surface growth [21,22]. Finally, it has already been shown through rigorous analysis and numerical experiments that these systems exhibit a very rich dynamical behavior, such as noise-induced transitions [23] and the possibility of stabilization of linearly

[1] Notice that the compactness of the resolvent of $\mathcal{A}$ implies that the operator has discrete spectrum which, together with the self-adjointness of $\mathcal{A}$ and the assumption that it commutes with the covariance operator of the noise Q, allow to expand the solution of (1) in terms of the eigenfunctions of $\mathcal{A}$.

[2] Usually the functions f and g involve derivatives of the function u. For example, for both the Burgers and the Kuramoto–Shivashinsky equation we have $f(u) = u\partial_x u$. The linear function $g(u)$ is included to induce a linear instability to the dynamics. In the case of the Burgers equation we will simply take $g(u) = u$ whereas in the case of the Kuramoto–Shivashinsky equation we can take $g(u) = \partial_x^4 u$. Further discussion can be found in Section 4 and in [20].