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Sparse tensor spherical harmonics approximation in radiative transfer

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ABSTRACT

The stationary monochromatic radiative transfer equation is a partial differential transport equation stated on a five-dimensional phase space. To obtain a well-posed problem, boundary conditions have to be prescribed on the inflow part of the domain boundary.

We solve the equation with a multi-level Galerkin FEM in physical space and a spectral discretization with harmonics in solid angle and show that the benefits of the concept of sparse tensor products, known from the context of sparse grids, can also be leveraged in combination with a spectral discretization. Our method allows us to include high spectral orders without incurring the "curse of dimension" of a five-dimensional computational domain.

Neglecting boundary conditions, we find analytically that for smooth solutions, the convergence rate of the full tensor product method is retained in our method up to a logarithmic factor, while the number of degrees of freedom grows essentially only as fast as for the purely spatial problem. For the case with boundary conditions, we propose a splitting of the physical function space and a conforming tensorization. Numerical experiments in two physical and one angular dimension show evidence for the theoretical convergence rates to hold in the latter case as well.

1. Introduction

1.1. Radiative transfer

In this work we address the numerical solution of the stationary monochromatic radiative transfer problem [see e.g. 1] defined on a bounded Lipschitz domain $D \subset \mathbb{R}^d$, where $d = 2, 3$.

We would like to find the radiative intensity $u(\mathbf{x},\mathbf{s}), u : D \times \mathcal{S}^{d_\mathcal{S}} \to \mathbb{R},\ \mathcal{S}^{d_\mathcal{S}}$ being the sphere with $d_\mathcal{S} = 1, 2$, that satisfies for all $(\mathbf{x},\mathbf{s}) \in D \times \mathcal{S}^{d_\mathcal{S}}$ the *radiative transfer equation* (RTE)

$$\mathbf{s}\cdot\nabla_x u(\mathbf{x},\mathbf{s}) + (\kappa(\mathbf{x}) + \sigma(\mathbf{x}))u(\mathbf{x},\mathbf{s}) = \kappa(\mathbf{x})I_b(\mathbf{x}) + \sigma(\mathbf{x})\int_{\mathcal{S}^{d_\mathcal{S}}} \Phi(\mathbf{s},\mathbf{s}')u(\mathbf{x},\mathbf{s}')d\mathbf{s}', \tag{1a}$$

and on the inflow boundary the boundary condition

$$u(\mathbf{x},\mathbf{s}) = g(\mathbf{x},\mathbf{s}), \quad \mathbf{x} \in \partial D, \quad \mathbf{s}\cdot\mathbf{n}(\mathbf{x}) < 0. \tag{1b}$$

Together, (1a) and (1b) form the *radiative transfer problem*. In this problem statement, $\kappa \geqslant 0$ is the absorption coefficient, $\sigma \geqslant 0$ the scattering coefficient, $I_b \geqslant 0$ the blackbody intensity, $\Phi \geqslant 0$ the scattering phase function, $g \geqslant 0$ the radiation

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entering the domain or wall emission, and $\mathbf{n}(\mathbf{x})$ the outer unit normal on the boundary. The scattering phase function is normalized to $\int_{\mathcal{S}^{d_\mathcal{S}}} \Phi(\mathbf{s},\mathbf{s}')\mathrm{d}\mathbf{s}' = 1$ for each direction $\mathbf{s}$, which corresponds to elastic scattering. From here on, we assume $g = 0$, i.e. the domain boundaries are non-emissive or "cold".

An introduction to the topic of radiative heat transfer is given by Modest [1]. Apart from Monte Carlo methods, standard solution approaches to the radiative transfer problem are the discrete ordinates method and the method of spherical harmonics. Frank [2] gives an overview of these numerical methods for radiative transfer. State-of-the-art methods and applications are compiled by Kanschat et al. [3].

In the discrete ordinate method or S_N-approximation, Eq. (1) is solved for N fixed directions spanning the full range in solid angle. The method is simple to implement and thus popular, but in order to capture very localized features of the solution in the $\mathbf{s}$-dependence a fine angular resolution is necessary. Also, the method suffers from so-called *ray effects*, in which the mesh structure of the discretization is reflected in the solution [4].

In the method of spherical harmonics or P_N-approximation, the intensity is expanded into a truncated series of spherical harmonics in solid angle, resulting in a coupled system of PDEs in space. Often used is the P_1-approximation, in which (1a) is reduced to a diffusion equation. In general, though, higher orders lead to a sharp increase in mathematical complexity when boundary conditions are to be satisfied [5]. This impediment may be a reason why one rarely finds higher orders than $N = 7$ or 15 used in practice. For smooth solutions, the spherical harmonics method exhibit spectral convergence, which makes them a popular and promising approach for radiative transfer problems where smoothness in the solution is expected when absorption or scattering are present.

The system of partial differential equations arising from the S_N- or P_N-approximation is discretized with finite differences or a finite element method. Manteuffel et al. [6], for instance, solve a least squares formulation with spherical harmonics in the solid angle and finite elements in space. Kanschat [7] combines the discrete ordinate method with a stabilized streamline diffusion FEM in the physical domain.

All these methods suffer from the "curse of dimension", the low rate of convergence in terms of number of degrees of freedom due to the high dimensionality of the radiative transfer problem, which is stated in five dimensions for $(d, d_\mathcal{S}) = (3, 2)$. The accuracy of the solution does not scale in the same way as the computational complexity so that accurate discretizations quickly become prohibitively expensive.

For the spherical harmonics method, there exist approaches to reduce the workload while maintaining accuracy. A simplified P_N-approximation is available [e.g. 8] by expanding the inverse of the transport operator to a given order. Modest and Yang [5] suggest a successive elimination of spherical harmonic tensors to reduce the number of simultaneous differential equations from $(N + 1)^2$ to $N(N + 1)/2$ in 3D. While these methods reduce the computation compared to the standard full radiative transfer problem, the overall asymptotic complexity remains the same. Furthermore, the analytical derivation of formulas to ensure satisfaction of the boundary conditions still becomes increasingly involved with higher order N.

Widmer et al. [9] have developed a method to overcome the curse of dimension in the context of a wavelet discretization of the angular domain. In their sparse tensor product method, they discretize physical and angular domain with hierarchical and wavelet finite elements, respectively, and then select only the most relevant finite element product combinations to construct the search space for the solution. Provided that the absorption coefficient $\kappa(\mathbf{x})$ and blackbody intensity $I_b(\mathbf{x})$ are sufficiently smooth, their method achieves a log-linear complexity in the number of degrees of freedom while convergence rates deteriorate only by a logarithmic factor. Their method is also suited for the optically thin regime, i.e. for small κ.

Even though the concept of sparse tensorization has been introduced to the solution of the radiative transfer equation some years ago now, and the notion of sparse grids exists even longer [10], the problem of the high computational costs is still present and unaddressed in recent publications about radiative transfer (e.g. for discrete ordinates the article by Monard and Bal [11], or FEM in angle by Becker et al. [12]).

To promote the idea of sparse tensorization, we combine the sparse tensor product method with a spectral discretization involving spherical harmonics, as already suggested by Widmer et al. [13]. Our aim is to show that the advantages of sparse tensorization carry over from a hierarchical FEM in physical space and wavelet FEM in solid angle as by Widmer et al. [13] to a combination of hierarchical FEM and spectral approach, thereby breaking the "curse of dimension" for spectral discretizations.

Moreover, we aim at eliminating order-dependent complications in the problem formulation by our treatment of "cold wall" boundary conditions. Physical boundary functions are tensorized conformingly with geometry-adapted spectral angular basis functions to satisfy zero inflow boundary conditions in a strong sense. Together with sparse tensorization, this method makes it possible to include spherical harmonics of high order in the solution of the radiative transfer problem.

The paper presents an extended and revised version of a recent preprint [14] and is organized as follows: in Section 2 we reformulate the radiative transfer problem (1) into a variational problem with a least squares approach.

In Section 3 we describe our discretization of the variational problem. We apply a Galerkin ansatz to the variational problem and define our product combination basis functions of hierarchical linear functions in physical space and spherical harmonics as well as Legendre polynomials in angular space. We define the full and sparse tensor product search space without and with boundary conditions and derive and prove approximation properties for the case neglecting boundary conditions.

Section 4 underlines the analytical derivations with results from numerical experiments in which we compare the usual full tensor product method to the sparse tensor product method.