Journal of Computational Physics 230 (2011) 8619–8646

Contents lists available at SciVerse ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

An efficient method for the incompressible Navier–Stokes equations on irregular domains with no-slip boundary conditions, high order up to the boundary

D. Shirokoff *, R.R. Rosales

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

ARTICLE INFO

Article history:
Received 29 November 2010
Received in revised form 11 July 2011
Accepted 12 August 2011
Available online 23 August 2011

Keywords:
Pressure Poisson equation
Poisson boundary conditions
Staggered grid
Incompressible flow
Projection methods
Navier–Stokes

ABSTRACT

Common efficient schemes for the incompressible Navier–Stokes equations, such as projection or fractional step methods, have limited temporal accuracy as a result of matrix splitting errors, or introduce errors near the domain boundaries (which destroy uniform convergence to the solution). In this paper we recast the incompressible (constant density) Navier–Stokes equations (with the velocity prescribed at the boundary) as an equivalent system, for the primary variables velocity and pressure. equation for the pressure. The key difference from the usual approaches occurs at the boundaries, where we use boundary conditions that unequivocally allow the pressure to be recovered from knowledge of the velocity at any fixed time. This avoids the common difficulty of an, apparently, over-determined Poisson problem. Since in this alternative formulation the pressure can be accurately and efficiently recovered from the velocity, the recast equations are ideal for numerical marching methods. The new system can be discretized using a variety of methods, including semi-implicit treatments of viscosity, and in principle to any desired order of accuracy. In this work we illustrate the approach with a 2-D second order finite difference scheme on a Cartesian grid, and devise an algorithm to solve the equations on domains with curved (non-conforming) boundaries, including a case with a non-trivial topology (a circular obstruction inside the domain). This algorithm achieves second order accuracy in the L^∞ norm, for both the velocity and the pressure. The scheme has a natural extension to 3-D.

1. Introduction

A critical issue in the numerical solution of the incompressible Navier–Stokes equations is the question of how to implement the incompressibility constraint. Equivalently, how to recover the pressure from the flow velocity, given the fact that the equations do not provide any boundary condition for the pressure. This has been an area of intense research, ever since the pioneering MAC scheme [15] of Harlow and Welch in 1965. Of course, one can avoid the problem by simultaneously discretizing the momentum and the divergence free equations, as in the difference scheme proposed by Krzywicki and Ladyzhenskaya [26], which can be shown to converge – while avoiding the need for any pressure boundary conditions. Approaches such as these, however, do not lead to efficient schemes.

Generally the dilemma has been that of a trade-off between efficiency, and accuracy of the computed solution near the boundary. However, many applications require both efficiency, and accuracy. For example, to calculate fluid solid

* Corresponding author.
E-mail addresses: shirokof@math.mit.edu (D. Shirokoff), rrr@math.mit.edu (R.R. Rosales).

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doi:10.1016/j.jcp.2011.08.011

interactions, both the pressure and gradients of the velocity are needed at the solid walls, as they appear in the components of the stress tensor. Furthermore, these objectives must be achievable for "arbitrary" geometries, not just simple ones with symmetries that can be exploited. Unfortunately, these requirements are not something that current algorithms are generally well suited for, as the brief review below is intended to show.[1] However, we believe that algorithms based on a pressure Poisson equation (PPE) reformulation of the Navier–Stokes equations – reviewed towards the end of this introduction – offer a path out of the dilemma. The work presented in this paper is, we hope, a contribution in this direction.

Projection methods are very popular in practice because they are efficient. They achieve this efficiency by (i) interpreting the pressure as effecting a projection of the flow velocity evolution into the set of incompressible fields. That is, write the equations in the form $\mathbf{u}_t = \mathcal{P}(\mu\Delta\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{f})$, where $\mathcal{P}$ is the appropriate projection operator, μ is the kinematic viscosity, $\mathbf{u}$ is the flow velocity vector, and $\mathbf{f}$ is the vector of applied body forces. (ii) Directly evolving the flow velocity. The question is then how to compute $\mathcal{P}$.

In their original formulation by Chorin [6] and Temam [45], the projection method was formulated as a time splitting scheme in which: First an intermediate velocity is computed, ignoring incompressibility. Second, this velocity is projected onto the space of incompressible vector fields – by solving a Poisson equation for pressure. Unfortunately this process introduces numerical boundary layers into the solution, which can be ameliorated (but not completely suppressed) for simple geometries – e.g. ones for which a staggered grid approach can be implemented [7].

The development of *second order projection methods* [2,22,24,32,47] provided greater control over the numerical boundary layers and accuracy in the pressure [4]. These are the most popular schemes used in practice. However, particularly for moderate or low Reynolds numbers, the effects of the numerical boundary layers can still be problematic [13]. Non-conforming boundaries add an extra layer of difficulty. The search for means to better control these numerical artifacts is an active area of research.

The numerical boundary layers in projection methods reflect in the known convergence results for them (e.g. [37,39,42]). Convergence is stated only in terms of integral norms, with the main difficulties near the boundary. There point-wise convergence (and even less convergence of the flow velocity gradient) cannot be guaranteed – even if the solution is known to be smooth. Hence the accurate calculation of wall stresses with these methods is problematic. Guermond et al. [13] provide further details on convergence results, as well as an extensive review of projection methods and the improved pressure-correction schemes.

Two other methods for solving the Navier–Stokes equations are the *immersed boundary* [28,30,34,35,44], and the *vortex-streamfunction* [3,5,31] methods. These also decouple the calculation of the velocity and of the pressure. The immersed boundary method does so by introducing Dirac forces to replace the domain walls, which makes obtaining high order implementation of the boundary conditions difficult. The vortex-streamfunction formulation decouples the equations, but has dimensional limitations. An interesting variation of the vortex-streamfunction approach, using only local boundary conditions, is presented in reference [14].

Closely related to the immersed boundary methods are the *penalty* (alternatively: *fictitious domain* or *domain embedding*) methods – e.g. see [1,9,23]. These methods, effectively, replace solid walls in the fluid by a porous media with a small porosity $0 < \eta \ll 1$. In the limit $\eta \to 0$, this yields no slip and no flow-through at the solid walls. Two important advantages of this approach are that complicated domains are easy to implement, and that the total fluid-solid force can be computed using a volume integral, rather than an integral over the boundary of the solid. Unfortunately, the parameter η introduces $\sqrt{\eta}$ boundary layers which make convergence slow and high accuracy computations expensive, since η cannot be selected independently of the numerical grid size.

Finally, we mention the algorithms based on a *pressure Poisson equation (PPE) reformulation of the Navier–Stokes equations* [16–18,29,12,20,21,25,38,40,41], which is the class of methods within which the work presented in this paper falls. In this approach the incompressibility constraint for the flow velocity is replaced by a Poisson[2] equation for the pressure. This then allows an extra boundary condition – which must be selected so that, in fact, incompressibility is maintained by the resulting system. This strategy was first proposed by Gresho and Sani [12], who pointed out that adding $\nabla\cdot\mathbf{u} = 0$ as a boundary condition yields a system of equations that is equivalent to the Navier–Stokes equations. Unfortunately, their particular PPE formulation incorporates no explicit boundary condition that can be used to recover the pressure from the velocity, by solving a Poisson problem – for a more detailed discussion of this, see Remark 4 in this paper. In [16,17] the issue is resolved at the discrete numerical level, where they demonstrate high order schemes. For instance [16] demonstrates a fourth-order in space and second-order in time implementation using overlapping grids. Subsequent work at the continuum level was later introduced by Henshaw and Anders Petersson [18] and Johnston and Liu [21]. Recently, work in PPE formulations have led to interesting improvements and analysis of projection methods [29]. In this paper – in Eqs. (20) and (21) – we present another PPE system, also equivalent to the Navier–Stokes equations, which allows an explicit recovery of the pressure given the flow velocity. A comparison with the one in [21] can be found in Remark 5 in this paper. Subtle issues can arise with semi-implicit approaches (pressure treated explicitly and viscous term implicitly) leading to time step restrictions [36]. In Section 5 we show that semi-implicit implementations of our scheme do not have time step restrictions of the diffusive type.

[1] This is not intended as a thorough review of the field, and we apologize for the many omissions.

[2] The choice of the Poisson equation for the pressure is not unique, e.g. see [20].