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Directional integration on unstructured meshes via supermesh construction

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ABSTRACT

Unstructured meshes are in widespread use throughout computational physics, but calculating diagnostics of simulations on such meshes can be challenging. For example, in geophysical fluid dynamics, it is frequently desirable to compute directional integrals such as vertical integrals and zonal averages; however, it is difficult to compute these on meshes with no inherent spatial structure. This is widely regarded as an obstacle to the adoption of unstructured mesh numerical modelling in this field. In this paper, we describe an algorithm by which one can exactly compute such directional integrals on arbitrarily unstructured meshes. This is achieved via the solution of a problem of computational geometry, constructing the supermesh of two meshes. We demonstrate the utility of this approach by applying it to a classical geophysical fluid dynamics system: the thermally driven rotating annulus. This addresses an important objection to the more widespread use of unstructured mesh modelling.

1. Introduction

Much scientific computing involves the approximate solution of a system of partial differential equations describing a physical problem of interest. These computations often rely on a discretisation of the geometry into a mesh upon which the approximation may be computed. There are two kinds of meshes: structured and unstructured. A structured mesh is a mesh for which nodal neighbourhood relationships do not need to be explicitly stored – i.e. simple nodal neighbourhood relationships exist for all nodes in the mesh. This is usually achieved by imposing that (quote) “all interior mesh nodes have an equal number of adjacent elements” [22]. An unstructured mesh is one which is not structured. As generating a structured mesh imposes additional topological constraints over generating an unstructured mesh, unstructured meshing is significantly easier, and is better suited to complex geometries [28] and mesh adaptivity techniques [21]. One can also generalise the concept of structure in a mesh: for example, it is possible to generate meshes that are horizontally unstructured, but are structured in the vertical direction so that all vertices are vertically aligned. Such meshes are often used in unstructured ocean modelling (e.g. [1,3,10,8,29,30]).

Heretofore, one strong criticism of the use of unstructured meshes in solving partial differential equations is the difficulty of computing relevant diagnostics of the simulation [2,27]. Typically, the analyst is not interested in the overall picture of the prognostic variables, but some diagnostic function of them: often such diagnostics are trivial to compute on meshes with a particular structure, but difficult to compute on general unstructured meshes. For example, it is often highly desirable to compute directional integrals of the prognostic quantities. In an ocean modelling context, vertical averaging is key to the

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determination of advective transports, as a proxy for measurements of upwelling or downwelling, and for computing depth-integrated stream functions. If the mesh is vertically structured, this computation is easy; if no vertical structure exists, such as in the fully unstructured ocean model presented by Pain et al. [23] and Piggott et al. [26], then this computation is very difficult.

One may choose to interpolate a model solution onto a mesh with an appropriate mesh structure for directional integrals. For example, in order to compute a vertical average, one may simply interpolate from the model solution mesh onto a vertically structured mesh. However, it is not clear that the interpolation errors introduced by such a procedure need be small. Indeed, in a simulation in which dynamic mesh adaptivity is applied, it is possible that small-scale features, resolved by the model, will be entirely missed by an interpolation procedure, and hence be absent from the model diagnostics. Significant effort may be invested in choosing the simulation mesh, and it is troubling if this is discarded in order to compute offline diagnostics. One may choose an intermediate structured diagnostic mesh to be of higher resolution in order to mitigate this problem. However, in a complex three-dimensional simulation, it may not be clear what resolution is required in order for accurate diagnostics to be obtained. At the very least, it is desirable for the error introduced to be quantifiable.

In this paper, we apply a technique of computational geometry to the computation of difficult directional integrals on unstructured meshes. The concept of a *supermesh* of two meshes was introduced in George and Borouchaki [9], and the first general algorithm for its efficient computation for arbitrarily unrelated meshes was given in Farrell and Maddison [6] (henceforth FM11). The supermesh is the mesh of the intersections of the elements of the input meshes (Fig. 1). The Galerkin projection of prognostic fields, via supermesh construction, was tested for applications in dynamic mesh adaptive ocean modelling in Hiester et al. [12], and was used to develop a geostrophic balance preserving interpolant in Maddison et al. [18].

In this paper, we apply supermeshing in a novel way to compute difficult diagnostics of quantities on unstructured meshes. The general strategy is to construct the supermesh of two input meshes: the unstructured mesh upon which our differential equation has been solved, and a mesh which has whatever structure that makes the computation of the relevant diagnostic easy. We then observe that in many cases the supermesh of the two inherits this particular structure, making the computation of the diagnostic straightforward on the supermesh. Furthermore, since the supermesh provides a function superspace of the two input meshes [5], interpolation from either unstructured mesh to the supermesh is lossless (subject to roundoff), provided basis functions are chosen on the supermesh that form a superspace for the function spaces defined on each of the input meshes (see Farrell [5] for a discussion). Therefore, the model output is interpolated losslessly onto the supermesh, where the diagnostic may be easily computed (and exactly computed, subject to roundoff). Essentially, the supermeshing procedure produces a new mesh, on which the diagnostic can be computed exactly. All of this is performed with a local implementation of supermesh construction: the entire supermesh is never held in memory all at once, only small parts at a time.

For the applications described in this paper the supermesh is used to perform the Galerkin projection of directionally averaged equations. The supermesh is used to compute inner products between basis functions defined on each of two input meshes. Note that, because of this very specific purpose, questions of mesh quality are generally irrelevant: no equations are solved on the supermesh itself. Rather, supermeshing is a tool which is used here to compute certain integrals accurately.

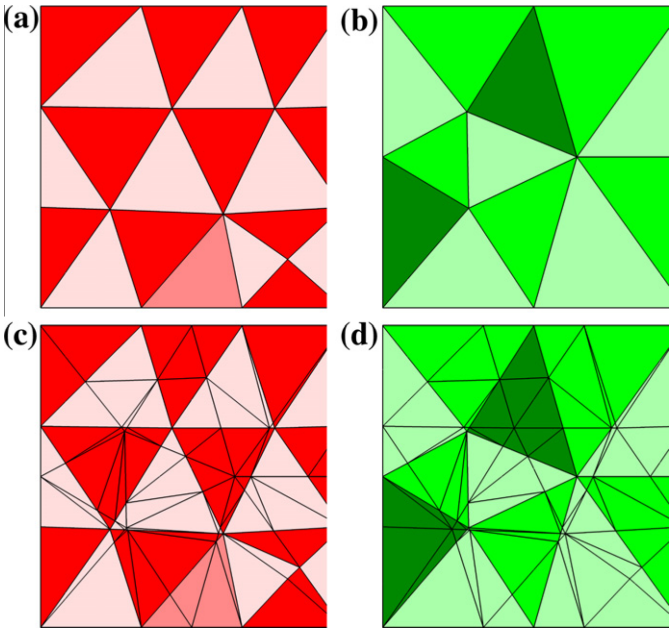

Fig. 1. (a, b) Two triangular meshes. (c) A triangular supermesh of (a) and (b), coloured to show the elements of (a). (d) The same supermesh of (a) and (b), coloured to show the elements of (b).