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Multi-level Monte Carlo finite volume methods for nonlinear systems of conservation laws in multi-dimensions

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ABSTRACT

We extend the multi-level Monte Carlo (MLMC) in order to quantify uncertainty in the solutions of multi-dimensional hyperbolic systems of conservation laws with uncertain initial data. The algorithm is presented and several issues arising in the massively parallel numerical implementation are addressed. In particular, we present a novel load balancing procedure that ensures scalability of the MLMC algorithm on massively parallel hardware. A new code is described and applied to simulate uncertain solutions of the Euler equations and ideal magnetohydrodynamics (MHD) equations. Numerical experiments showing the robustness, efficiency and scalability of the proposed algorithm are presented.

1. Introduction

A number of problems in physics and engineering are modeled in terms of systems of conservation laws:

$$\begin{cases} \mathbf{U}_t + \operatorname{div}(\mathbf{F}(\mathbf{U})) = 0, \quad \forall (\mathbf{x}, t) \in \mathbf{D} \times \mathbb{R}_+, \\ \mathbf{U}(\mathbf{x}, 0) = \mathbf{U}_0(\mathbf{x}), \end{cases} \tag{1.1}$$

Here, $\mathbf{U} : \mathbf{D} \subset \mathbb{R}^d \mapsto \mathbb{R}^m$ denotes the vector of conserved variables and $\mathbf{F} : \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathbb{R}^{m \times d}$ is the collection of directional flux vectors. The partial differential equation is augmented with initial data $\mathbf{U}_0$.

Examples for conservation laws include the shallow water equations of oceanography, the Euler equations of gas dynamics, the magnetohydrodynamics (MHD) equations of plasma physics and the equations of non-linear elasticity.

It is well known that solutions of (1.1) in general develop discontinuities or shock waves in finite time even for smooth initial data. Hence, solutions of (1.1) are sought in the sense of distributions, see [10]. Furthermore, additional admissibility criteria or *entropy conditions* are imposed to ensure uniqueness.

As the equations are non-linear, analytical solution formulas are only available in very special situations. Consequently, numerical schemes are the main tools for the study of systems of conservation laws. Many efficient numerical schemes for approximating systems of conservation laws are currently available. They include the finite volume, conservative finite difference and discontinuous Galerkin methods, see [21,14]. Alternatives include the front tracking method, see [17].

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Existing numerical methods for approximating (1.1) require the initial data $\mathbf{U}_0$ as the input. However, in most practical situations, it is not possible to measure this input precisely. The measurement of other inputs like sources, boundary data and coefficients is also prone to uncertainty. This uncertainty in the inputs for (1.1) results in the propagation of uncertainty in the solution. The modeling and approximation of the propagation of uncertainty in the solution due to uncertainty in inputs constitutes the theme of uncertainty quantification (UQ).

Uncertainty in inputs and solutions of PDEs is frequently modeled in a probabilistic manner. The inputs are random fields with prescribed probability laws. The solution is also realized as a random field and the statistical moments of the solutions like the expectation and variance are the quantities of interest.

It is highly non-trivial to develop efficient algorithms for quantifying uncertainty in conservation laws. The biggest challenge lies in the fact that the discontinuities in physical space may lead to the propagation of discontinuities in the probability space. A robust numerical method should be able to deal with these discontinuities. Another challenge lies in dealing with the fact that the number of random sources driving the uncertainty may be very large (possibly infinite).

The design of efficient numerical schemes for quantifying uncertainty in solutions of conservation laws has seen a lot of activity in recent years. The most popular methods are the stochastic Galerkin methods based on generalized polynomial chaos (gPC for short). An incomplete list of references on gPC methods for uncertainty quantification in hyperbolic conservation laws includes [3,7,22,32,27,33] and other references therein. Although these *deterministic* methods show some promise, they suffer from the disadvantage that they are highly *intrusive*. Existing codes for computing deterministic solutions of conservation laws need to be completely reconfigured for implementation of the gPC based stochastic Galerkin methods. Furthermore, some of the intrusive schemes appear rather difficult to parallelize. An alternative class of methods for quantifying uncertainty in PDEs are the stochastic collocation methods, see [36] for a general review and [25,35] for modifications of these methods near discontinuities. Stochastic collocation methods are non-intrusive and easier to parallelize than the gPC based stochastic Galerkin methods.

Another class of methods for computational uncertainty quantification in numerical solutions of PDEs are statistical sampling methods, most notably Monte Carlo (MC) sampling. In a MC method, the probability space is *sampled* and the underlying deterministic PDE is solved for each sample. The MC samples of numerical solutions of the PDE are combined into statistical estimates of expectation and other statistical moments of the random solution which are necessary to quantify uncertainty. In uncertainty quantification for hyperbolic scalar conservation laws with random initial data, MC type methods together with finite volume (FV) spatio-temporal discretizations of the PDE were proposed in a recent paper [26]. The MC-FVM methods were analyzed in the context of a scalar conservation law with random initial data and corresponding estimates of the combined discretization and statistical sampling errors were obtained. MC methods are non-intrusive and therefore are very easy to code. Existing deterministic PDE solvers are reused in MC codes. As it was shown in [26], MC methods converge at rate 1/2 as the number M of MC samples increases. The asymptotic convergence rate $M^{-1/2}$ is non-improvable by the central limit theorem. Therefore, MC methods require a large number of "samples" (with each "sample" involving the numerical solution of (1.1) with a given draw of initial data $\mathbf{U}_0$) in order to ensure low statistical errors. This slow convergence entails high computational costs for MC type methods. In particular, quantifying uncertainty with MC methods for systems of conservation laws in several space dimensions with moderately high number of sources of uncertainty becomes very costly.

In order to address this drawback of the MC methods, we proposed a novel *multi-level Monte Carlo* (MLMC) algorithm for *scalar* conservation laws in [26]. Multi-level MC methods were introduced by Heinrich for numerical quadrature [16] and developed by Giles to enhance the efficiency of path simulations for Itô stochastic ordinary differential equations in [11,12]. More recently, MLMC finite element methods for elliptic problems with stochastic coefficients were introduced by Barth et al. in [4]. MLMC methods for SPDEs with applications have been recently proposed in [5,8,13].

In [26], we presented and analyzed MLMC-FVM for scalar conservation laws with *random initial data*. Based on our asymptotic error analysis, we derived in [26] an optimized combination of sampling sizes on different levels of spatial and temporal resolution to achieve maximum accuracy in the statistical estimates of first and higher order moments of the random solution. We proved that for first order FV solvers of the SCL, the MLMC-FVM obtained in this way allows the computation of approximate statistical moments with the *same accuracy versus cost ratio as a single deterministic solve on the same mesh*. While offering dramatically improved efficiency over standard MC methods, the MLMC-FV methods developed in [26] are still totally non-intrusive and are as easy to code and parallelize as traditional, single level MC FV methods. Numerical examples in one space dimension showed that MLMC-FVM was orders of magnitude faster than standard MC-FVM.

Our main aim in this paper is to extend the MLMC-FVM algorithm to systems of conservation laws in several space dimensions. As a convergence theory for numerical schemes approximating such systems of conservation laws is currently out of reach, even for the deterministic problem, we use the error estimates derived in [26] as a guide to choose the number of samples per mesh level in the MLMC algorithm. The absence of rigorous error estimates makes it imperative to test the MLMC-FVM for systems of conservation laws for a large number of benchmark problems and demonstrate the efficiency of this algorithms for systems in several space dimensions. We do so in this paper.

As compared to [26], in the present paper several interesting implementation issues are addressed: due to the massive computational effort entailed by the accurate numerical solution of multi dimensional systems of conservation laws, numerical solves of (1.1) for single samples of random initial data $\mathbf{U}_0$ can not be performed on a single processor. Therefore, in the present paper the MLMC-FVM algorithm is extended towards several forms of parallelism: parallel computation of large