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Conservative and non-conservative methods based on Hermite weighted essentially non-oscillatory reconstruction for Vlasov equations

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ABSTRACT

We develop weighted essentially non-oscillatory reconstruction techniques based on Hermite interpolation both for semi-Lagrangian and finite difference methods. We apply these methods to transport equations in the context of plasma physics and the numerical simulation of turbulence phenomena. On the one hand the non-conservative semi-Lagrangian methods with high order reconstructions are particularly efficient and accurate in linear phase of simulations before the appearance of small structures. However in the nonlinear phase, the lack of conservations may generate inaccurate numerical simulations. At contrast, the conservative finite difference methods are more stable in nonlinear phase and the Hermite WENO reconstruction avoids spurious oscillations.

1. Introduction

Turbulent magnetized plasmas are encountered in a wide variety of astrophysical situations like the solar corona, accretion disks, etc., but also in magnetic fusion devices such as tokamaks. In practice, the study of such plasmas requires solving the Maxwell equations coupled to the computation of the plasma response. Different ways are possible to compute this response: the fluid or the kinetic description. Unfortunately, the fluid approach seems to be insufficient when one wants to study the behavior of zonal flow, the interaction between waves and particles or the occurrence of turbulence in magnetized plasmas, for example. Most of the time these plasmas are weakly collisional, and then they require a kinetic description represented by the Vlasov–Maxwell system. The numerical simulation of the full Vlasov equation involves the discretization of the six-dimensional phase space $(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$, which is still a challenging issue. In the context of strongly magnetized plasmas however, the motion of the particles is particular since it is confined around the magnetic field lines; the frequency of this cyclotron motion is faster than the frequencies of interest. Therefore, the physical system can be reduced to four or five dimensions by averaging over the gyroradius of charged particles (see a review in [1,14]).

Development of accurate and stable numerical techniques for plasma turbulence (4D drift kinetic, 5D gyrokinetic and 6D kinetic models) is one of our long-term objectives. Of course, there are already a large variety of numerical methods based on direct numerical simulation techniques. The Vlasov equation is discretized in phase space using either semi-Lagrangian

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[8,9,28,29], finite element [17], finite difference [30] or discontinuous Galerkin [6,19] schemes. Most of these methods are based on a time splitting discretization which is particularly efficient for classical Vlasov–Poisson or Vlasov–Maxwell systems since the characteristic curves corresponding to the split operator simply become straight lines and can be solved exactly. Therefore, the numerical error is only due to the splitting in time and the phase space discretization of the distribution function. Furthermore for such time splitting schemes, the semi-Lagrangian methods on Cartesian grids coupled with Lagrange, Hermite or cubic spline interpolation techniques are conservative [3,9]. Hence, these methods are currently used and have proven their efficiency for various applications and in this context they are often observed to be less dissipative than classical finite volume or finite difference schemes. However, for more elaborated kinetic equations like the 4D drift kinetic [15] or 5D gyrokinetic [16] equations, or even the two-dimensional guiding center model [29], time splitting techniques cannot necessarily be applied. Thus characteristic curves are more sophisticated and required a specific time discretization. For instance, in [15,16] several numerical solvers have been developed using an Eulerian formulation for gyro-kinetic models. However, spurious oscillations often appear in the nonlinear phase when small structures occur and it is difficult to distinguish physical and numerical oscillations. Moreover, for these models semi-Lagrangian methods are no more conservative, hence the long time behavior of the numerical solution may become unsuitable.

For this purpose, we want to develop a class of numerical methods based on the Hermite interpolation which is known to be less dissipative than Lagrange interpolation [9], together with a weighted essentially non-oscillatory (WENO) reconstruction applied to semi-Lagrangian and finite difference methods. Actually, Hermite interpolation with WENO schemes was already studied in [25] in the context of discontinuous Galerkin methods with slope limiters. A system of equations for the unknown function and its first derivative are evolved in time and used in the reconstruction. Moreover, a similar technique, called CIP (Cubic Interpolation Propagation), has also been proposed for transport equations in plasma physics applications [23], but the computational cost is strongly increased since the unknown and all the derivatives are advected in phase space. In [9], a semi-Lagrangian method with Hermite interpolation has been proposed and shown to be efficient and less dissipative than Lagrangian interpolation. In this method, the first derivatives are approximated by a fourth-order centered finite difference formula.

Here, we also apply a similar pseudo-Hermite reconstruction [9] and meanwhile introduce an appropriate WENO reconstruction to control spurious oscillation leading to nonlinear schemes. We develop third and fifth order methods and apply them to semi-Lagrangian (non-conservative schemes) and conservative finite difference methods. Our numerical results will be compared to the usual semi-Lagrangian method with cubic spline interpolation [29] and the classical fifth-order WENO finite difference scheme [21].

The paper is organized as follows. We first present the Vlasov equation and related models which will be investigated numerically. Then in Section 3, the semi-Lagrangian method is proposed with high order Hermite interpolation with a WENO reconstruction to control spurious oscillations. In Section 4, conservative finite difference schemes with Hermite WENO reconstructions are detailed. In Section 5, a discussion of approximation of first derivatives is presented. Then the one-dimensional free transport equation with oscillatory initial data is investigated to compare our schemes with classical ones (semi-Lagrangian with cubic spline interpolation and conservative finite difference schemes with WENO reconstruction). Finally we perform numerical simulations on the simplified paraxial Vlasov–Poisson model and on the guiding center model for highly magnetized plasma in two dimensions.

2. The Vlasov equation and related models

The evolution of the density of particles $f(t, \mathbf{x}, \mathbf{v})$ in the phase space $(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^d \times \mathbb{R}^d$, $d = 1, .., 3$, is given by the Vlasov equation,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathbf{F}(t, \mathbf{x}, \mathbf{v}) \cdot \nabla_{\mathbf{v}} f = 0, \tag{2.1}$$

where the force field $F(t, \mathbf{x}, \mathbf{v})$ is coupled with the distribution function f giving a nonlinear system. We mention the well known Vlasov–Poisson (VP) model describing the evolution of particles under the effects of self-consistent electro-magnetic fields. We define the charge density $\rho(t, x)$ by

$$\rho(t, \mathbf{x}) = q \int_{\mathbb{R}^d} f(t, \mathbf{x}, \mathbf{v})\, d\mathbf{v}, \tag{2.2}$$

where q is the single charge. The force field is given for the Vlasov–Poisson model by

$$\mathbf{F}(t, \mathbf{x}, \mathbf{v}) = \frac{q}{m} \mathbf{E}(t, \mathbf{x}), \qquad \mathbf{E}(t, \mathbf{x}) = -\nabla_{\mathbf{x}} \phi(t, \mathbf{x}), \qquad -\Delta_{\mathbf{x}} \phi = \frac{\rho}{\varepsilon_0}, \tag{2.3}$$

where m represents the mass of one particle. These equations and related reduced equations, such as 4D drift-kinetic equation [18], are frequently used to describe plasma turbulence in a tokamak core.