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A multi-moment vortex method for 2D viscous fluids

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ABSTRACT

In this paper we introduce simplified, exact, combinatorial formulas that arise in the vortex interaction model found in [33]. These combinatorial formulas allow for the efficient implementation and development of a new multi-moment vortex method (MMVM) using a Hermite expansion to simulate 2D vorticity. The method naturally allows the particles to deform and become highly anisotropic as they evolve without the added cost of computing the non-local Biot–Savart integral. We present three examples using MMVM. We first focus our attention on the implementation of a single particle, large number of Hermite moments case, in the context of quadrupole perturbations of the Lamb–Oseen vortex. At smaller perturbation values, we show the method captures the shear diffusion mechanism and the rapid relaxation (on $Re^{1/3}$ time scale) to an axisymmetric state. We then present two more examples of the full multi-moment vortex method and discuss the results in the context of classic vortex methods. We perform numerical tests of convergence of the single particle method and show that at least in simple cases the method exhibits the exponential convergence typical of spectral methods. Lastly, we numerically investigate the spatial accuracy improvement from the inclusion of higher Hermite moments in the full MMVM.

1. Introduction

A new viscous vortex interaction model was introduced in [33,43] whose main purpose is to study the dynamics of the viscous n-vortex problem. As a first application, the model was shown in [32], to improve the approximation of the far field acoustic pressure field generated by a pair of rotating vortices, when compared to classical point vortex methods. One of the strengths of the model is the reduction of the 2D vorticity equation to a system of ODE's with purely quadratic nonlinearity which represent the evolution of the Hermite moments of each vortex. The main drawback of the model as presented in [33] is that the coefficients in the ODE's are computed in terms of derivatives and limits of an explicit kernel function which are computationally intractable for higher resolution (more Hermite moments) computations.

We introduce in Appendix A simplified, combinatorial exact formulas for the coefficients of these terms, greatly reducing the computational costs of generating and implementing the ODEs. These simplified formulas allow the viscous vortex interaction model of [33] to be implemented as a *new* multi-moment vortex method to simulate the two-dimensional vorticity equation. The novelty of this multi-moment vortex method (MMVM) is that the vortex particles are allowed to dynamically deform under the flow, *without* the usual difficulties of computing the Biot–Savart kernel for anisotropic basis elements, thus reducing the convection error associated to classic vortex methods.

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It is important to note that the multi-moment vortex method we propose here shares many features with traditional vortex methods for incompressible [2,7,12] and compressible [15] flows which are comprehensively reviewed in [14]. One of the advantages that this method shares with traditional vortex methods is that it is gridless. In particular, we can avoid the common computational issues of using a large grid approximation for an unbounded domain. Using vortex methods, one also avoids the necessity of imposing either artificial, or periodic boundary conditions which may introduce unwanted or unphysical effects. MMVM also naturally incorporates the effects of viscosity in a way most closely related to the core spreading method, utilized in the work of Leonard [22], Rossi [37], Barba and Leonard [3], and Huang et al. [19]. Core spreading, using a purely Lagrangian vortex method, was originally shown to converge to the wrong solution in [17]. An adaptive method, proposed by Rossi [37], has been shown to fix this convergence problem and since then, innovative interpolation and initialization methods [5], have been developed to more accurately address this convergence problem.

To incorporate anisotropic deformations, our method uses a Hermite moment expansion for each vortex element. For two-dimensional *inviscid* fluids, Melander et al. [27,28] also developed a model based on the evolution of moments of vortices. Instead of using Hermite moments, Melander et al. define a local coordinate system at each vortex and compute equations for the evolution of the centroid along with the time evolution of the local *geometric* moments. While in theory equations may be computed to any order for these expansions relatively low order approximations are used in [27,28] and convergence issues are not addressed. Using a Hermite moment expansion is natural, as they form a suitable basis and conditions for global in time convergence have been proven for MMVM [33]. In addition, it is assumed in [27,28] that the maximum diameter of any localized vortex is much smaller than the minimum distance between any two vortices. This would specifically not allow the use of many overlapping vortices to simulate the fluid dynamics, i.e. a vortex method approach. Nonetheless, Melander et al. are able to show, using well separated vortices with moments up to second order, significant improvement over the point vortex method for several computed examples involving well-separated, co-rotating vortices.

To our knowledge, the most closely related work to MMVM is Rossi's deformable vortex method using elliptically deforming Gaussian vortex particles. Rossi implemented the method first for the advection diffusion equation [38,39] and also for the Navier–Stokes equations [40,35]. For both the advection diffusion equation and the Navier–Stokes equations, Rossi's deformable vortex method was able to achieve higher order convergence, as compared to non-deforming vortices, but only if one also modifies the Lagrangian velocity field. Unfortunately, the method was hampered by the difficulty of computing the Biot–Savart law for anisotropic (elliptical) basis functions to reconstruct the velocity field. While several ideas were proposed to alleviate this problem [40,35], we note that one of the most promising advantages of our approach is the avoidance of this difficult computation altogether, as the Biot–Savart integral is explicitly known to all orders in our Hermite expansion.

For general flows the primary source of error from vortex methods arises from convection error which is caused by advecting vortices *without deformation*. Thus our method should greatly reduce this error since it allows for many different deformations, limited only by the order of the Hermite approximation. In fact, constructing basis functions that are allowed to deform have been conjectured as precisely what is need to improve the spatial accuracy [6] of vortex methods as seen already by Rossi's improvement from second to fourth order with elliptical deformed Gaussian basis functions [40]. Prior to Rossi's work elliptically deforming basis elements were used by Moeleker and Leonard [31] in a filtered advection–diffusion equation. In [31] the authors use a Hermite expansion but only to compute an approximation to the average velocity field, and not the vorticity field itself. This modification to the Lagrangian velocity field did not result in an increase of the order of accuracy.

In this paper we will first analyze the single-particle MMVM ($n = 1$) with a large number of Hermite moments, m. Unlike usual particle methods, our single vortex element must be large and model an entire vortex structure. As a first application of the single-particle MMVM, we consider the problem of shear diffusion around a rotationally symmetric vortex. The shear diffusion manifests itself in this context through the mixing of vorticity and the axisymmetrization of the vorticity profile on a $Re^{1/3}$ time scale [8]. MMVM is a natural choice to study this problem for several reasons. As we will discuss in Section 2 our single-particle MMVM is equivalent to implementing a Hermite spectral method. Thus, while offering the usual good approximation properties of classical spectral methods it avoids the artificial periodic boundary conditions associated to classical spectral methods (Fourier, Chebyshev), see [9]. Aside from these general advantages, the single-particle MMVM offers some particular advantages for the study of two-dimensional fluid motion. Theoretical results have shown that the Hermite functions we use for our expansion give a natural basis with respect to which one can study the long-time asymptotics of solutions of the two-dimensional vorticity equation [16]. Furthermore, the vorticity distribution (in contrast to the fluid velocity) has the property that if it is initially localized it will remain so for later times. This means that if our Hermite expansion of the initial vorticity distribution converges, it will do so for all later times [33].

We next consider two examples using the full MMVM: a model for early time vortex merger behavior using $n = 2$ vortex elements, and second, we consider tripole relaxation using a course grid approximation $n = 36$. In both cases we observe increasingly accurate solutions as we increase the number of moments m. MMVM has several important parameters to choose for implementation. Just as in regular vortex methods, one must choose the number of particles n, and the overlap ratio which is computed by dividing the particle core size by the particle spacing. In addition, we must also specify the number of Hermite moments m for each particle to include in the simulation. As we will see both parameters n and m can be increased to achieve greater accuracy and, at least in the case of a single particle ($n = 1$) the spatial convergence is faster than polynomial in m.

We will work primarily with the 2D vorticity equation which can be derived from the Navier–Stokes equations for two dimensional incompressible fluids: