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Efficient numerical schemes for viscoplastic avalanches. Part 1: The 1D case

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ABSTRACT

This paper deals with the numerical resolution of a shallow water viscoplastic flow model. Viscoplastic materials are characterized by the existence of a yield stress: below a certain critical threshold in the imposed stress, there is no deformation and the material behaves like a rigid solid, but when that yield value is exceeded, the material flows like a fluid. In the context of avalanches, it means that after going down a slope, the material can stop and its free surface has a non-trivial shape, as opposed to the case of water (Newtonian fluid). The model involves variational inequalities associated with the yield threshold: finite-volume schemes are used together with duality methods (namely Augmented Lagrangian and Bermúdez–Moreno) to discretize the problem. To be able to accurately simulate the stopping behavior of the avalanche, new schemes need to be designed, involving the classical notion of well-balancing. In the present context, it needs to be extended to take into account the viscoplastic nature of the material as well as general bottoms with wet/dry fronts which are encountered in geophysical geometries. We derived such schemes and numerical experiments are presented to show their performances.

1. Introduction

The objective of this paper is to present improved numerical schemes for shallow water models for viscoplastic materials on variable topography (or bathymetry). The associated difficulties are twofold. First, we will place ourselves in a context – increasingly in use – where no regularization method is used and thus the variational inequalities reflecting the plastic nature of the material are handled directly through duality methods. Second, we will describe new well-balanced schemes in this viscoplastic context to take into account general bottoms *and* wet/dry fronts.

In recent years, an increasing interest has been developed for shallow water models in the context of simulations for the flow of viscoplastic materials down inclined planes. Viscoplastic materials are characterized by the existence of a yield stress: below a certain critical threshold in the imposed stress, there is no deformation and the material behaves like a rigid solid, but when that yield value is exceeded, the material flows like a fluid. Such flow behavior can be encountered in many practical situations such as food pastes, heavy oils, lavas and avalanches. As a consequence, the theory of the fluid mechanics of such materials has applications in a wide variety of fields such as chemical industry, energy industry and geophysical fluid dynamics.

From the mathematical viewpoint, the non-linearity associated to viscoplastic models (such as the Bingham model, as we will see below), leads to feasible but very expensive computational times for the full 3D equations (see e.g. [28]).

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Consequently, numerous reduced 2D model using the shallow flow approximation have been derived. In the context of avalanches, we refer to the article of Ancey [2] and references therein for a detailed review on these developments. Recently, in [25], an interesting shallow water model based on a Bingham-like constitutive law together with Coulomb frictional condition on the bottom was derived, in local coordinates for the case of a non-planar topography. But the algorithm, presented to solve associated equations, does not take into account either well-balanced properties or the treatment of wet/dry fronts. Another shallow model based on the Herschel–Bulkley constitutive law (which generalizes the Bingham law) was derived in [1] and a new well-balanced scheme was introduced to take into account both non-linearities of this constitutive law, leading to a scheme which preserves more accurately stationary states. This point is important when it comes to determine the stopping time of the flow, when the material enters in its rigid state. And this kind of property is also linked to the use of duality methods which allow to properly deal with the plasticity.

Indeed, a common point of an increasing part of the recently developed numerical methods for viscoplastic flows (see e.g. [33,37,36,24,35,25]) is that they use decomposition-coordination methods to solve the variational inequality associated to constitutive laws with a so-called plastic threshold (the most simple and iconic one being the Bingham model). This kind of approach takes its roots in the seminal works of Duvaut–Lions [15] and the series of papers of Glowinski and coworkers (see the recent book [22] for a detailed review), initiated at the end of the seventies and anchored in the Augmented Lagrangian formalism. One of the crucial advantage of these methods over regularized approaches (see e.g. Papanastasiou's [29] or the so-called bi-viscosity methods [14]) is that they rigorously take into account the plastic threshold. Of note, it is well known that in Augmented Lagrangian (AL) methods, the optimal values of the parameters are not easy to determine in the general case. These parameters ((r, ρ) in Glowinski's nomenclature) influence the speed of convergence of the iterative process towards the saddle-point, solution of the problem. As a consequence, a study of some sort of optimality for such parameters is of real interest when it comes to improve the computational efficiency.

As an alternative duality method, one can use the so-called Bermúdez–Moreno (BM) algorithm. This method, which is built upon some properties of the Yosida regularization of maximal monotone operators, has been extensively used for a wide range of applications (see [19] and the references therein). In order to apply the method, the Yosida regularization of the subdifferential associated to the non-differentiable operator appearing in the formulation of the considered model needs to be determined. As for the AL, the performance of the algorithm strongly depends on the choice of two constant parameters. Fortunately, several ways to overcome this problem have been proposed in the literature [31,30,19], and they will be considered in this paper.

Another difficulty that appears when it comes to couple shallow-water models and viscoplastic constitutive laws, is the adequate *coupling* between the discretization associated to the duality method and the one associated with the spatial terms, *in such a way the global scheme is well-balanced*. For shallow water type equations, finite volume methods have proved their efficiency and we adopt them in the present work. In this context, a careful treatment must be made to design a *well-balanced* scheme when coupling the finite volume scheme and the duality method. This idea was first introduced in [8], in the context of a Bingham fluid treated with an AL method. We extend here this idea for a Shallow-Water–Bingham model on a general topography and in the presence of wet/dry fronts.

The well-balanced properties are related to the stationary solutions of the system. In our case, we seek numerical schemes which preserve exactly two types of stationary solutions. For hyperbolic systems with source terms, a discretization of the source terms compatible with the one of the flux term must be performed. Otherwise, stemming from the numerical diffusion terms, a first order error in space takes place. This error, after time iteration, may yield large errors in wave amplitude and speed. The pioneering work by Roe [32] relates the choice of the approximation of the source term with the property of preserving stationary solutions. Bermúdez and Vázquez Céndón introduced in [5] an extension of Roe's solver, in the context of *shallow water equations*, which preserves exactly the stationary solution of water at rest. This work originated the so-called *well-balanced solvers*, in the sense that the discrete source terms balance the discrete flux terms when computed on some (or all) of the steady solutions of the continuous system. Different extensions have been done: see for instance Greenberg and Leroux [23], LeVeque [27], Chacón et al. [12].

An additional difficulty in the simulation of free surface flows comes from the appearance of dry areas in the computational domain, due to the fluid evolution or to the initial conditions. Standard numerical schemes may compute spurious solutions in the presence of wet/dry fronts, unless appropriate modifications are made. See [7,34] for a review on some methods appearing in the literature to deal with this problem. Moreover, in the context of shallow water equations, Roe-type schemes lose their well-balanced properties when wet/dry transitions appear. Indeed, they may produce nonphysical negative values of the thickness of the water layer near the wet/dry front. Some ways to modify Roe's method to fix these problems have been proposed in [10,11].

The contribution of this paper is twofold. First, by adapting the guidelines in [30], we determine, in a theoretical way, an optimal choice of parameters in the sense that they provide the highest rate of convergence for the BM algorithm. For the AL, we perform some numerical studies of the optimal choice of parameters and we then compare both methods on various problems to give insight on their respective behaviors. To our best knowledge this is the first time that BM is applied to such kind of models and that such a systematic comparative study of the behavior of the number of iteration in duality methods is done for several very different viscoplastic flows.