Journal of Computational Physics 264 (2014) 133–151

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

A robust multilevel method for hybridizable discontinuous Galerkin method for the Helmholtz equation

Huangxin Chen [a,*], Peipei Lu [b,c], Xuejun Xu [c]

[a] School of Mathematical Sciences, Xiamen University, Xiamen, China
[b] School of Mathematics Sciences, Soochow University, Suzhou, 215006, China
[c] LSEC, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, P.O. Box 2719, Beijing, 100190, China

ARTICLE INFO

Article history:
Received 24 June 2013
Received in revised form 25 January 2014
Accepted 27 January 2014
Available online 31 January 2014

Keywords:
Multilevel method
Helmholtz equation
High wave number
Hybridizable discontinuous Galerkin method
GMRES method
Local Fourier analysis

ABSTRACT

A robust multilevel preconditioner based on the hybridizable discontinuous Galerkin method for the Helmholtz equation with high wave number is presented in this paper. There are two keys in our algorithm, one is how to choose a suitable intergrid transfer operator, and the other is using GMRES smoothing on the coarse grids. The multilevel method is performed as a preconditioner in the outer GMRES iteration. To give a quantitative insight of our algorithm, we use local Fourier analysis to analyze the convergence property of the proposed multilevel method. Numerical results show that for fixed wave number, the convergence of the algorithm is mesh independent. Moreover, the performance of the algorithm depends relatively mildly on wave number.

1. Introduction

In this paper we consider the Helmholtz equation with Robin boundary condition which is the first order approximation of the radiation condition. The equation is written in a mixed form as follows: Find $(\boldsymbol{q}, u)$ such that

$$\mathbf{i}\kappa \boldsymbol{q} + \nabla u = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$\mathbf{i}\kappa u + \operatorname{div} \boldsymbol{q} = f \quad \text{in } \Omega, \tag{1.2}$$

$$-\boldsymbol{q} \cdot \boldsymbol{n} + u = g \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal or polyhedral domain, $\kappa > 0$ is known as the wave number, $\mathbf{i} = \sqrt{-1}$ denotes the imaginary unit, and $\boldsymbol{n}$ denotes the unit outward normal to $\partial\Omega$. The Helmholtz equation finds applications in many important fields, e.g., in acoustics, seismic inversion and electromagnetic, but how to solve the Helmholtz equation efficiently is still of great challenge.

The strong indefiniteness has prevented the standard multigrid methods from being directly applied to the discrete Helmholtz equation. In [9], Elman, Ernst and O'Leary modified the standard multigrid algorithm by adding GMRES iterations as corrections on coarse grids and using it as an outer iteration. But in order to obtain a satisfactory convergence behavior, a relatively large number of GMRES smoothing should be performed on coarse grids which leads to relatively large memory requirement, so the optimality of the multigrid algorithm cannot be guaranteed. In [6], the authors utilized the continuous

* Corresponding author.
E-mail addresses: chx@xmu.edu.cn (H. Chen), pplu@suda.edu.cn (P. Lu), xxj@lsec.cc.ac.cn (X. Xu).

0021-9991/$ – see front matter
http://dx.doi.org/10.1016/j.jcp.2014.01.042

interior penalty finite element methods [23,24] to construct the stable coarse grid correction problems, which reduces the steps of GMRES smoothing on coarse grids. Based on the fact that the error components which cannot be reduced by the standard multigrid can be factorized by representing it as the product of a certain high-frequency Fourier component and a ray function, Brandt and Livshits introduced so-called wave-ray multigrid methods in [3,20]. Although this method exhibits high convergence rate with increasing wave number, it does not easily generalize to unstructured grids and complicated Helmholtz problems. Besides, shifted Laplacian preconditioners [13,14] and sweeping preconditioners [10,11] based on an approximate LDL^t factorization were introduced to solve the Helmholtz equation with high wave number. A survey of the development of fast iterative solvers can be found in [12,15].

Hybridizable discontinuous Galerkin (HDG) method has two main advantages in the discretization of Helmholtz equation. First, it is a stable method, which means that the discrete system is always well-posed without any mesh constraint. Rigorous convergence analysis of the HDG method for the Helmholtz equation can be found in [5]. Second, comparing to standard discontinuous Galerkin method, HDG method results in significantly reducing the degrees of freedom, especially when the polynomial degree p is large. However, to the best of our knowledge, no efficient iterative method or preconditioner for HDG discretization system for the Hemholtz equation in the literature has been proposed.

The hybridized system is a linear equation for Lagrange multipliers which is obtained by eliminating the flux as well as the primal variable. For the second-order elliptic problems, a Schwarz preconditioner for the algebraic system was presented in [18]. In [19], the authors consider the application of a variable V-cycle multigrid algorithm for the hybridized mixed method for second-order elliptic boundary value problems. In their multigrid algorithm, both smoothing and correction on coarse grids are based on standard piecewise linear continuous finite element discretization system. The convergence of the multigrid algorithm depends on an assumption that the number of smoothing increases in a specific way (see Theorem 3.1 in [19] for details). The critical ingredient in the algorithm is how to choose a suitable intergrid transfer operator. Numerical experiments in [19] show that certain 'obvious' transfer operators lead to slow convergence.

The objective of this paper is to propose a robust multilevel method for the HDG approximation of the Helmholtz equation. The main ingredients in the multilevel method are how to construct the coarse grid correction problem and perform efficient smoothing. Since strong indefiniteness arises for the Helmholtz equation with large wave number, standard Jacobi or Gauss–Seidel smoothers become unstable on the coarse grids. Motivated by the idea in [9], we use GMRES smoothing for those coarse grids. Unlike the smoothing strategy in [9], the number of GMRES smoothing steps in our algorithm is much smaller, even if one smoothing step may guarantee the convergence of our multilevel algorithm. Moreover, smoothing on both fine and coarse grids in our multilevel method is based on hybridized system of Lagrange multiplier on each level.

Local Fourier analysis (LFA) has been introduced for multigrid analysis by Achi Brandt in 1977 (cf. [2]). We mainly utilize the LFA to analyze smoothing properties of relaxations and convergence properties of two and three level methods in the one dimensional case. This may provide quantitative insights into the proposed multilevel method for the Helmholtz problem (1.1)–(1.3). A survey for LFA can be found in [22].

The remainder of this paper is organized as follows: In Section 2, we firstly review the formulation of HDG method for the Helmholtz equation and present our multilevel algorithm. The stability estimate of the intergrid transfer operator will be carried out in Section 3. Section 4 is devoted to the LFA of the multilevel method in one dimensional case. Finally, we give some numerical results to demonstrate the performance of our multilevel method.

2. HDG method and its multilevel algorithm

Let $\mathcal{T}_h$ be a quasi-uniform regular subdivision of Ω, and denote the collection of edges (faces) by $\mathcal{E}_h$, while the set of interior edges (faces) by $\mathcal{E}_h^0$ and the collection of element boundaries by $\partial\mathcal{T}_h := \{\partial T \mid T \in \mathcal{T}_h\}$. We define $h_T := \operatorname{diam}(T)$ and let $h := \max_{T\in\mathcal{T}_h} h_T$. Throughout this paper we use the standard notations and definitions for Sobolev spaces (see, e.g., Adams [1]).

On each element T and each edge (face) F, we define the local spaces of polynomials of degree $p \geqslant 1$:

$$\boldsymbol{V}(T) := \left(\mathcal{P}_p(T)\right)^d, \qquad W(T) := \mathcal{P}_p(T), \qquad M(F) := \mathcal{P}_p(F),$$

where $\mathcal{P}_p(S)$, $S = T$ or F, denotes the space of polynomials of total degree at most p on S. The corresponding global finite element spaces are given by

$$\begin{aligned}
\boldsymbol{V}_h^p &:= \left\{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \;\middle|\; \boldsymbol{v}|_T \in \boldsymbol{V}(T) \text{ for all } T \in \mathcal{T}_h\right\},\\
W_h^p &:= \left\{w \in L^2(\Omega) \;\middle|\; w|_T \in W(T) \text{ for all } T \in \mathcal{T}_h\right\},\\
M_h^p &:= \left\{\mu \in L^2(\mathcal{E}_h) \;\middle|\; \mu|_F \in M(F) \text{ for all } F \in \mathcal{E}_h\right\},
\end{aligned}$$

where $\boldsymbol{L}^2(\Omega) := (L^2(\Omega))^d$, $L^2(\mathcal{E}_h) := \prod_{F\in\mathcal{E}_h} L^2(F)$. On these spaces we define the bilinear forms

$$(\boldsymbol{v}, \boldsymbol{w})_{\mathcal{T}_h} := \sum_{T\in\mathcal{T}_h} (\boldsymbol{v}, \boldsymbol{w})_T, \qquad (v, w)_{\mathcal{T}_h} := \sum_{T\in\mathcal{T}_h} (v, w)_T, \quad \text{and} \quad \langle v, w\rangle_{\partial\mathcal{T}_h} := \sum_{T\in\mathcal{T}_h} \langle v, w\rangle_{\partial T},$$

with $(\boldsymbol{v}, \boldsymbol{w})_T := \int_T \boldsymbol{v}\cdot\boldsymbol{w}\,dx$, $(v, w)_T := \int_T vw\,dx$ and $\langle v, w\rangle_{\partial T} := \int_{\partial T} vw\,ds$.