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Short Note

A simple mass conserving semi-Lagrangian scheme for transport problems ☆

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ABSTRACT

A mass conserving semi-Lagrangian (SL) scheme is achieved with a combination of a simple explicit smoothness-based mass correction and a standard non-conservative interpolating SL scheme. The resulting mass correction can be incorporated into any existing SL scheme with negligible extra cost. A more selective and less damping monotonicity filter by comparison to traditional filters is also presented. Results from various tests from the literature show that, in addition to mass conservation, the proposed scheme has negligible impact on the overall accuracy of the standard non-conservative SL scheme.

1. Introduction

Semi-Lagrangian (SL) advection schemes are widely used in atmospheric models. Their unconditional stability and the computational efficiency concomitant from using large time steps have been the main reasons for their use as transport schemes for many weather and climate models [1]. The main drawback of SL schemes is the lack of mass conservation for quantities where conservative transport is crucial to the accuracy and the appropriate physical behaviour of the model.

Mass conservation has been achieved using either mass fixing schemes or conservative remappings. Recently there have been a number of conservative remapping schemes developed to inherently conserve mass [2–4], including the SLICE algorithm [5]. Multi-dimensional remappings can be relatively expensive due to the extra geometric computations needed and this has been the main reason for their limited application in operational models. On the other hand, mass fixing schemes [6–11], whereby global mass conservation is restored diagnostically, are more appealing due to their simplicity and the relatively negligible cost of incorporating them within existing SL schemes. Moreover, the loss of mass using standard SL schemes, especially high-order schemes, is relatively small. The combination of the relative cheapness of the method and the small mass deficit makes mass fixers a viable and attractive proposition for mass conservation within SL schemes. However, the main criticism of these fixers is the ad hoc nature of where the deficit/surplus is added or subtracted.

The goal of this paper is to present a simple and efficient scheme for restoring mass conservation to the standard non-conservative SL solution without a significant impact on the accuracy of the original SL solution. Although, conservative remappings are more mathematically based and probably describe more accurately the evolution of local and global mass integrals, the present scheme can be an efficient alternative for the circumstances where mass conservation is paramount yet the computational resources are limited. The rest of the paper is organised as follows: Section 2 briefly describes a standard SL scheme and the proposed mass fixing scheme together with a more improved monotonicity filter; results using the

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proposed scheme are presented in Section 3 and compared with those using standard non-conservative SL; and conclusions are summarised in Section 4.

2. Semi-Lagrangian transport

Consider a passive advection of a scalar quantity governed, in the absence of sources, by

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0, \tag{1}$$

where $\rho(\mathbf{x}, t)$ is the density, at location $\mathbf{x}$ at time t, of the transported quantity; $\mathbf{u}$ is the fluid velocity; and $D/Dt \equiv \partial/\partial t + \mathbf{u} \cdot \nabla$ is the Lagrangian derivative following the fluid.

Integrating (1) over a finite volume element $V(t)$ moving with the fluid leads to the classical Lagrangian integral form of the tracer conservation:

$$\frac{D}{Dt}\left(\int_{V(t)} \rho dV \right) = 0. \tag{2}$$

Eq. (2) implies that the mass contained in an element $V(t)$ that moves with the fluid, is invariant in time. A consequence of (2) is that in the absence of sources/sinks the global mass is also invariant for a periodic domain (e.g., sphere) or a closed problem (i.e., zero fluxes at the boundaries). Any point $\mathbf{x}$ of the volume $V(t)$ moves according to the trajectory equation

$$\frac{D\mathbf{x}}{Dt} = \mathbf{u}. \tag{3}$$

Given the discrete solution $\rho_i^n \equiv \rho(\mathbf{x}_i, t^n)$ at N Eulerian (fixed) grid points $\{\mathbf{x}_i, i = 1,\ldots,N\}$ and the known velocity field $\mathbf{u}(\mathbf{x}_i, t)$, the solution of (1) (i.e., $\rho_i^{n+1} \equiv \rho(\mathbf{x}_i, t^{n+1} = t^n + \Delta t)$), using a two-time-level, central, semi-implicit, semi-Lagrangian scheme, can be written as:

$$\rho_i^{n+1} - \rho_{i,d}^n = -\frac{\Delta t}{2}\rho_i^{n+1}(\nabla \cdot \mathbf{u})_i^{n+1} - \frac{\Delta t}{2}\rho_{i,d}^n(\nabla \cdot \mathbf{u})_{i,d}^n, \tag{4}$$

where the subscripts (i,d) refers to evaluation (or interpolation) at the location $\mathbf{x}_{i,d}$, which is the departure point of $\mathbf{x}_i$, determined from the integration of (3) (i.e., $\mathbf{x}_i - \mathbf{x}_{i,d} = \int_{t^n}^{t^{n+1}} \mathbf{u}(\mathbf{x}, t)dt$). Expressing terms at the departure point in terms of known counterparts at grid-points, (4) can be rewritten as:

$$\rho_i^{n+1} = \beta_i \sum_{j=1}^{N} \alpha_{ij} \rho_j^n, \tag{5}$$

where α_{ij} are interpolation weights and β_i accounts for the averaging of the divergence along the trajectory,

$$\beta_i = \frac{2 - \Delta t(\nabla \cdot \mathbf{u})_{i,d}^n}{2 + \Delta t(\nabla \cdot \mathbf{u})_i^{n+1}} = \frac{2 - \Delta t \sum_{j=1}^{N} \alpha_{ij} (\nabla \cdot \mathbf{u})_j^n}{2 + \Delta t(\nabla \cdot \mathbf{u})_i^{n+1}}, \tag{6}$$

where $\beta_i = 1$ for a non-divergent flow (i.e., $\nabla \cdot \mathbf{u} = 0$).

Evaluation of a quantity at a departure point means using an interpolation from the known field at the Eulerian grid $\{\mathbf{x}_i, i = 1,\ldots,N\}$. Often there is also the requirement to interpolate different fields at the same location (e.g., transport of multi-species in chemistry models) and therefore it is more efficient to compute the interpolation weights α_{ij} then compute the interpolated values for the different species as simple summations as in (5). Although the present scheme does not impose any restriction on α_{ij} (including weights derived from global interpolants such as splines), here only Lagrange polynomials weights are used, as they are commonly used with SL schemes. SL schemes often use Lagrangian polynomials up to a certain degree p (i.e., $\alpha_{ij} \neq 0$ for only $H = (p + 1)^k$ points surrounding the interpolation point $\mathbf{x}_{i,d}$, where $k = 1, \ldots, 3$ is the dimension of the problem).

2.1. Mass conservation

It is clear that the interpolation operation (5) does not guarantee mass conservation. Let us assume that the non-conservative high-oder SL solution obtained from (5) is rewritten as:

$$\tilde{\rho}_i^{n+1} = \beta_i \sum_{j=1}^{N} \alpha_{ij}^H \rho_j^n, \tag{7}$$

where α_{ij}^H are derived from a high-order interpolant, i.e., $\alpha_{ij}^H \neq 0$ for H points surrounding the interpolation target, $2^k < H \leqslant N$. A solution $\{\rho_i^{n+1},\ i = 1,\ldots,N\}$ is said to be mass conservative iff the following constraint:

$$\sum_{i=1}^{N} \rho_i^{n+1} V_i = \sum_{i=1}^{N} \rho_i^n V_i = \sum_{i=1}^{N} \rho_i^0 V_i \equiv M_0, \tag{8}$$