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Perfectly matched layers for the heat and advection–diffusion equations

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ABSTRACT

We design a perfectly matched layer for the advection–diffusion equation. We show that the reflection coefficient is exponentially small with respect to the damping parameter and the width of the PML and this independently of the advection and of the viscosity. Numerical tests assess the efficiency of the approach.

1. Introduction

We are concerned here with the problem of truncating domains to compute numerical solutions of problems in unbounded domains so that the solution of the problem in the reduced domain is a good approximation to the solution of the original problem. In their seminal work on the wave equation, Engquist and Majda [1] introduced a quite general technique to address this problem by designing absorbing boundary conditions (ABC). Their technique has been applied to various equations and systems of equations in many fields: acoustics, electromagnetism, fluid dynamics elastodynamics and so on. As far as the heat equation is concerned, in [2–5], ABCs are designed at the continuous level and in [6] at the discrete level. In all these works, the difficulty lies in the approximation of the square root of a partial differential operator by a partial differential operator. This problem is inherent to the application of the procedure in [1] to the heat operator. Let us mention also the use of analytic solution with fast Fourier transforms, see [7] and references therein.

For hyperbolic equations such as the wave or Maxwell equations, a different way to handle artificial boundaries was introduced by Berenger [8,9]. In this method, the computational domain is surrounded by a dissipative and non reflexive artificial media (perfectly matched layer, PML). There is no need to approximate the square root of an operator by a partial differential operator.

Since then, many works have been devoted to a better understanding of their principle and behavior see [10–20] to extensions to other geometries, see [21,22], or equations see [23–27]. In these works, the equations are hyperbolic and the need for a PML comes from the propagative modes that exist in the solution. For propagative equations, the purpose of a PML is to turn a propagative mode into a vanishing one.

In this paper, we consider a parabolic equation for which there are only vanishing modes. We show that it is nevertheless possible to design and test a PML for the advection–diffusion equation:

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$$\mathcal{L}(u) := \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} - \nu\Delta u \quad (1)$$

including the heat equation as a special case. In our work, the purpose of the PML is to turn a slowly vanishing mode into a rapidly vanishing one. This result was announced as a short note in [28] where a brief presentation of the fundamental ideas was given. Here we provide more details and also prove that the efficiency is not dependent on the parameters of the equation such as the viscosity or the convection, see Eq. (8) and its proof. We also provide extensive numerical results that assess the validity of the PML for parabolic equations.

The paper is organized as follows. In the first part, we analyze the operator (1) in the Fourier space and introduce the perfectly matched layers to the advection–diffusion equation. In the second part, we apply this method to numerical computation to validate our approach.

2. Perfectly matched layers

2.1. Fourier analysis of the operator $\mathcal{L}$

In order to study the operator $\mathcal{L}$, we look for solutions of the equation $\mathcal{L}(u) = 0$ and make use of the Fourier transform. Let $u(t,x,y)$ be a function and $\hat{\hat{u}}(\omega, x, k)$ be its Fourier transform w.r.t. the variables t and y and let $\mathcal{F}^{-1}$ denote the inverse Fourier transform. We have:

$$\left(i\omega + a\partial\_x + bik - \nu\partial_{xx} + \nu k^2\right)(\hat{\hat{u}}(\omega, x, k)) = 0$$

For fixed ω and k, this is an ordinary differential equation in the variable x whose solutions are of the form

$$\hat{\hat{u}}(\omega, x, k) = \alpha(\omega, k)\exp\lambda^+(\omega, k)x + \beta(\omega, k)\exp\lambda^-(\omega, k)x$$

where

$$\lambda^{\pm}(\omega, k) := \frac{\frac{a}{\nu} \pm \sqrt{\frac{a^2}{\nu^2} + \frac{4}{\nu}(i\omega + ikb + \nu k^2)}}{2} \quad (2)$$

with $\mathcal{R}e\sqrt{z} \geqslant 0$. The coefficients α and β are fixed by the boundary conditions.

2.2. Definition of the PML equations

The operator $\mathcal{L}$ (see Eq. (1)) is originally defined in the whole plane $\mathbb{R}^2$ and we want to truncate the domain $x > 0$ by a PML. The PML model for this operator $\mathcal{L}$ is defined by replacing the x-derivative by a "*pml*" x-derivative. The definition is as follows. Let $\sigma > 0$ be a positive damping parameter, we define

$$\partial_x^{pml}(u) := \mathcal{F}^{-1}\left(\frac{i\omega + ikb}{i\omega + ikb + \frac{\nu}{4}\sigma}\partial_x\hat{\hat{u}}(\omega, x, k)\right) \quad (3)$$

and

$$\mathcal{L}_{pml} := \partial_t + a\partial_x^{pml} + b\partial_y - \nu\left(\partial_x^{pml}\right)^2 - \nu\partial_{yy} \quad (4)$$

be the PML equation with the following interface conditions at $x = 0$ between the solution u_{cd} in the convection–diffusion media and u_{pml} the solution in the PML media:

$$u_{cd} = u_{pml} \quad \text{and} \quad \partial_x(u_{cd}) = \partial_x^{pml}(u_{pml}) \quad (5)$$

2.3. Reflection coefficient

We show in this section that the reflection coefficient for a PML of width $\delta > 0$ is exponentially small with respect to the damping parameter σ and the width δ and this independently of the advection (a, b), the viscosity ν and the Fourier number, see formula (8). For this, we use the following setting which mimics the classical computation of the reflection coefficient for PML for the wave equation. The function

$$u_{inc} := \mathcal{F}^{-1}(\exp\lambda^-(\omega, k)x)$$

satisfies

$$\mathcal{L}(u_{inc}) = 0$$

and u_{inc} tends to zero as x tends to infinity. We approach this special solution by the following problem where the domain is truncated on the right by the PML: