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A matched interface and boundary method for solving multi-flow Navier–Stokes equations with applications to geodynamics

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ABSTRACT

We have developed a second-order numerical method, based on the matched interface and boundary (MIB) approach, to solve the Navier–Stokes equations with discontinuous viscosity and density on non-staggered Cartesian grids. We have derived for the first time the interface conditions for the intermediate velocity field and the pressure potential function that are introduced in the projection method. Differentiation of the velocity components on stencils across the interface is aided by the coupled fictitious velocity values, whose representations are solved by using the coupled velocity interface conditions. These fictitious values and the non-staggered grid allow a convenient and accurate approximation of the pressure and potential jump conditions. A compact finite difference method was adopted to explicitly compute the pressure derivatives at regular nodes to avoid the pressure–velocity decoupling. Numerical experiments verified the desired accuracy of the numerical method. Applications to geophysical problems demonstrated that the sharp pressure jumps on the clast-Newtonian matrix are accurately captured for various shear conditions, moderate viscosity contrasts and a wide range of density contrasts. We showed that large transfer errors will be introduced to the jumps of the pressure and the potential function in case of a large absolute difference of the viscosity across the interface; these errors will cause simulations to become unstable.

1. Introduction

This paper proposes a novel numerical method for solving the Navier–Stokes equations describing multi-flows, i.e., flows with distinct density and viscosity in subdomains of the flow field. Our study is motivated by the need to simulate highly viscous creeping flows that model a wide variety of geodynamic processes; these processes usually involve viscous, non-Newtonian visco-elastic, viscoplastic, or visco-elasto-plastic rheologies [2,6,19,30,32,37,39,51,54]. Of particular interest is the convection in Earth's mantle that occurs at depths ranging from about 100 km to 2900 km [7,21,33,40,61]. The rocks within Earth's mantle behave visco-plastically over geologic time scales (thousands to millions of years), and behave elasto-plastically over time scales associated with the earthquake cycle and seismic wave propagation (seconds to hundreds of years). The strength of mantle rocks varies with depth, with the elastic and viscous behaviors being different. The elastic moduli increase monotonically with depth, due primarily to the increasing pressure, with the shear modulus ranging approximately from 60 to 300 GPa and the bulk modulus ranging from about 100 to 600 GPa [1,9,15]. The viscous behavior is more complicated, with the viscosity decreasing with the depth above about 660 km. Below 660 km, there are conflicting

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models, some advocating a stepwise monotonic increase in the viscosity with depth [1] and some advocating a maximum viscosity located within the middle of the lower mantle [58]. It is generally agreed, however, that mantle viscosities range from 10^{18} to 10^{23} Pa s [37,38]. The pressure within the mantle varies from about 2 to 150 GPa and the density varies from 3100 to 5500 kg/m^3, both increasing with depth. Pressure-induced phase changes occur within the transition zone that separates the upper and lower mantle, between 660 and 900 km depth, making the changes in pressure, density, viscosity, and elastic moduli stepwise continuous within this interval. A dramatic change in earth properties occurs at the interface between the mantle and the outer core (approximately 2900 km depth). Here, the vertical derivative of the bulk modulus approximately doubles, the shear modulus drops to zero within the outer core, density increases by a factor of two, and viscosity decreases by 25 orders of magnitude [1,9,15].

Modeling mantle Stokes flow with these sharp viscosity contrasts (within the transition zone and at the core-mantle boundary) and over these wide ranges of viscosities poses challenges to geophysicists and mathematicians. In most geodynamically relevant cases this difficulty is further complicated by the need to track moving interfaces between different subdomains within the mantle (for example, upwelling plumes of geochemically or thermally distinct material). Standard numerical methods such as finite difference (FD), finite volume (FV), finite element (FE) and spectral methods have a long history of applications in modeling the mantle convection, as summarized in [60]. Recently, a number of specialized techniques have been developed to capture the sharp variation of viscosity across model domain boundaries. These include a multi-grid method [28], an adaptive multilevel wavelet collocation method [52], a hybrid spectral/finite difference method [41], and a finite element method with Q2P1 basis functions [42]. Some of these methods have been implemented in popular software for computational geophysics, such as GANGO [20], Gale [14], and CitCom [29,47]. However, many of these numerical methods do not have a sufficiently high resolution to resolve the sharp viscosity contrasts (which often occur over the width of a few numerical grid points) adequately to obtain accurate solutions. This is particularly a problem if a mesh element is cut by the material interface. A recent comprehensive study [8] shows that the quality of the solutions from either finite difference methods or finite element methods depends critically on the averaging technique used to define the viscosity at the interface elements, and on the type of mesh for finite element methods. The accuracy of the pressure solution appears more sensitive to the definition of the viscosity in numerical grids, and in some cases its error can be two orders of magnitude greater than the error in velocity. The convergence of the numerical solution is found slow except for the case where the material interface is perfectly fitting the mesh. These observations motivate us to introduce jumps of velocity, pressure and their derivatives induced by the discontinuous viscosity and density into the numerical simulations of the mantle convection on regular Cartesian grid. We anticipate that this novel numerical approach will not only improve the accuracy of the solution of the tectonic stress, but will also help save time for dynamically generating an interface-fitting numerical grid when the approach is combined with level set, volume-of-fluid (VOF) or other interface tracking techniques.

Solving fluid flow with an internal interface using finite difference methods on the regular Cartesian grid was first proposed by Peskin in simulating the blood flow through different valves in the heart [35]. The heart wall and valves interfacing atria and ventricles are described as elastic membranes with vanishing thickness moving with the fluid particles next to them. The forces exerted by the membranes to the blood flow are therefore singular in nature, and are spread to the grid points nearby through appropriately represented discrete delta functions. Immersed boundary methods are widely used in simulating fluid–solid interactions in biology fluid dynamics [36], including the most recent simulations of molecular motor proteins, microtubules and other subcellular organelles [3]. Instead of smoothing out the singular forces, the ghost flow method defines ghost points to carry the flow variables of the other fluid at the grid points physically occupied by a fluid. The jump conditions for the velocity and pressure at the interface are utilized to define the flow variables at the ghost points [12,13]. In many applications the variables of a real fluid are directly taken as the ghost variables if these variables are continuous across the interface, while the ghost values of the discontinuous variables are usually computed via extrapolation. These treatments limit the accuracy of the numerical approximation to the first order because a continuous variable of the flow field may have discontinuous derivatives at the interface. Third-order approximations to the interface conditions are constructed in the immersed interface method (IIM) by LeVeque and Li [23]. In IIM methods proper terms are devised to correct the standard central difference schemes for discretizing the derivatives near the interface so that a globally second-order solution can be obtained. The appealing mathematical features and many promising applications of the IIM methods to various types of differential equations are well documented by Li and Ito [24]. In addition to these and many other finite difference methods based on the Cartesian grid [53,59,63,62], finite volume methods and finite elements methods are developed to solve the interface problems on unstructured girds that are not necessarily conforming to the internal interface [25,11,27,16,34,17]. Specialized basis functions are needed to represent the kinks or the jumps of the solutions on the interface elements. Progress has been made in designing basis functions satisfying the interface conditions for solving single elliptical or parabolic equation, but much needs to be done to solve systems of differential equations with coupled interface conditions, for example, the Navier–Stokes equations.

The formulations and implementability of interface conditions of the Navier–Stokes equations differ significantly depending on the nature of the fluid flow. If the interface conditions are induced by the singular force exerted by the internal membrane, jumps of the velocity components and pressure are decoupled and can be given as functions of the force on the membrane only [26]. See [56] also for a systematic derivation of interface conditions for Navier–Stokes equations and [22] for the computation of the singular force and the consistent jumps of the velocity and pressure using a spline interpolation. For flows with distinct viscosities or densities in different subdomains, the continuity conditions for the viscous stress on the interface couple the velocity and pressure, and thus pose substantial difficulty for their numerical implementation