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Compact fourth-order finite volume method for numerical solutions of Navier–Stokes equations on staggered grids

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ABSTRACT

The development of a compact fourth-order finite volume method for solutions of the Navier–Stokes equations on staggered grids is presented. A special attention is given to the conservation laws on momentum control volumes. A higher-order divergence-free interpolation for convective velocities is developed which ensures a perfect conservation of mass and momentum on momentum control volumes. Three forms of the nonlinear correction for staggered grids are proposed and studied. The accuracy of each approximation is assessed comparatively in Fourier space. The importance of higher-order approximations of pressure is discussed and numerically demonstrated. Fourth-order accuracy of the complete scheme is illustrated by the doubly-periodic shear layer and the instability of plane-channel flow. The efficiency of the scheme is demonstrated by a grid dependency study of turbulent channel flows by means of direct numerical simulations. The proposed scheme is highly accurate and efficient. At the same level of accuracy, the fourth-order scheme can be ten times faster than the second-order counterpart. This gain in efficiency can be spent on a higher resolution for more accurate solutions at a lower cost.

1. Introduction

Over a half century, Computational Fluid Dynamics (CFD) plays an important part helping engineers and scientists to understand the nature of turbulent flows. An accurate time-dependent numerical simulation of turbulent flow can be obtained by direct numerical simulation (DNS) which solve the discrete Navier–Stokes equations directly. DNS can be very accurate but extremely expensive. The complexity of a DNS is roughly rising with $O(Re^{9/4})$. This scaling restricts DNS to low or moderate Reynolds numbers. A promising alternative simulation to DNS is the large eddy simulation (LES) in which the large-scale structures of the flow are resolved and the small-scale structures are modelled.

In an essence of numerical simulations, these two approaches rely heavily on the accuracy of the spatial information of the flow field. A satisfactory simulation cannot be obtained if the dynamics of the flow are not described in a sufficiently accurate way. Modelling effects of the small scales in an LES will not improve the overall accuracy of the solution when the numerical error was larger than the effects of the small scales [1]. The accuracy of the flow field information can be improved by increasing the numerical grid points or increasing the accuracy order of the numerical approximations. The latter approach has become an active field of research in recent years.

Higher-order approximations can be computed explicitly using Lagrange polynomials. The nth order approximation of the mth order derivative requires $n + m$ abscissas. Alternatively, one can couple unknown values to the abscissas and solve a system of linear equations. These implicit approximations have shorter stencils and have been called *compact scheme* by Lele [2]

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who demonstrates the superiority of compact schemes over traditional explicit schemes. At intermediate wave numbers, the compact fourth-order scheme is even better than the explicit sixth-order scheme. He quantified the resolution characteristics of second and higher-order schemes and showed that for a relative error of 0.1%, the fourth-order compact differentiation requires 5 points per half-wave, the fourth-order explicit requires 8 grid points and the second-order requires 50. In three-dimensional simulations, the total number of grid points grows cubically while the cost of higher-order schemes is linearly proportional to the second-order scheme. Thus using higher-order schemes is more attractive than a brute force increasing of resolution.

The finite volume methods (FVM) hold a strong position in CFD community because of their intrinsic conservation properties. Despite the popularity of the second-order FVM, there are only a few papers addressing its developments towards higher-order. The complicated relationship of volume-averaged values and surface fluxes made higher-order FVM more difficult than the finite difference (FD) counterpart. The first work that tries to link compact schemes to FVM is presented by Gaitonde and Shang [3]. They present fourth- and sixth-order compact finite volume methods for linear wave phenomena. However, the so-called reconstruction procedure is needed to compute the primitive value, which costs significant computational time. A more economical approach is proposed by Kobayashi [4]. He directly calculates the surface-averaged values from the volume-averaged ones. Explicit and implicit approximations based on the cell-averaged value up to 12th-order are analysed. Pereira et al. [5] present a compact finite volume method for the two-dimensional Navier–Stokes equations on collocated grids. Piller and Stalio [6] propose a compact finite volume method on staggered grids in two dimensions. Lacor et al. [7] develops a finite volume method on arbitrary collocated structured grids and performs LES of a turbulent channel flow at $Re_\tau = 180$. LES of the same flow with explicit filtering is performed in [8] using the spatial discretisation of [5]. Fourth-order finite volume in cylindrical domain is developed in [9] and a DNS of pipe flow at $Re_\tau = 360$ is performed.

Staggered grids have become a favourable arrangement over collocated grids because of the pressure decoupling problem. The pressure decoupling is not confined only in the second-order scheme. This problem is already reported in [5] when using even number of cells with a fourth-order scheme. Recently, a staggered grid has been shown to be more robust than collocated one by Nagarajan et al. [10] in large-eddy simulations. Thus compact finite volume methods on staggered grids deserves more attention.

On staggered grids, there are three problems to solve in order to achieve higher-order accuracy under finite volume discretisation, namely (i) *the approximation of convective velocities*, (ii) *the treatment of nonlinear terms* and (iii) *the discretisation of the pressure term*. The convective process requires the convective velocities on the surfaces of the momentum cells which are defined staggered to the pressure cells. On collocated grids, the face-averaged values of the momentum can be used to approximate the convective velocities because they are positioned correctly. However, on staggered grids the face-averaged momentums are aligned differently and the convective velocities must be interpolated accordingly. A simple interpolation, however, can violate the conservation of mass on the momentum cells. Higher-order divergence-free interpolations are still an open issue. The treatment of nonlinearity of the convective fluxes over the cell surface was addressed by Pereira et al. [5]. Nevertheless, the reconstruction of the nonlinear fluxes must be chosen wisely. The role of the pressure term in higher-order methods is still a matter of controversy among researchers in this field. It has been shown in [11–13] that the approximation of the pressure term has to be the same order as the one of the convective and diffusive fluxes. When the pressure is approximated using lower order, the overall accuracy is limited by this approximation. In [6,14] the second-order solution of pressure is found to be sufficient for a fourth-order accurate solution of velocities. This issue must be clarified because it is crucial to the cost of computations. The solution of the pressure can easily take more than half of the computation time. If a second-order approximation of the Poisson equation for the pressure was sufficient, then the higher-order accuracy of the momentum can be achieved at a marginal cost. However, if a fourth-order treatment of the pressure is necessary, a 19-point stencil of the Laplacian operator must be used instead of a simple 7-point stencil.

In the present work, we have two objectives. The first objective is to present the development of a fourth-order method for finite volume discretisation of the Navier–Stokes Equations by solving the questions posed in the previous paragraph. We propose a novel interpolation that preserves the discrete divergence-free property of the velocity fields. This method is generalized for arbitrary order of accuracy. Another fourth-order convective velocity that is not divergence-free is presented for comparison. Several choices of nonlinear corrections and the role of the pressure term are studied. We present new cell-centered deconvolutions for the mass and pressure fluxes. These approximations are explicit and lead to a banded system of the Poisson equation given by the projection method. The higher resolution properties of these cell-centered deconvolutions are demonstrated by the comparative Fourier analysis. We show that the solution of the pressure with lower order indeed limits the accuracy of the solution. However, this limitation on staggered grids is not as severe as on collocated ones reported in [11]. We use Fourier analysis to show that staggered grids can satisfy the incompressibility constraint in a better way than collocated grids and more information can be retained in the flow field.

The second objective of this work is to verify whether the fourth-order scheme can out perform the second-order scheme in terms of accuracy and efficiency. Despite the fact that higher-order schemes are shown to be vastly superior to second-order schemes in laminar flows by numerous authors, some recent papers report disappointing findings in the application of higher-order schemes to turbulent flows. Gullbrand [15] applies the fully-conservative explicit fourth-order scheme of Morinishi et al. [16] and Vasilyev [17] to a DNS of turbulent channel flow. Knikker [13] developed a compact finite difference method and applied it to the same flow. The grid resolutions used in their simulations are comparable to those used by the spectral code in [18]. They both report that differences between the results from second-order and fourth-order schemes are negligible and these results are significantly different from the spectral code. Meinke et al. [19] comment that the sixth-order