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Approximate linear response for slow variables of dynamics with explicit time scale separation

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ABSTRACT

Many real-world numerical models are notorious for the time scale separation of different subsets of variables and the inclusion of random processes. The existing algorithms of linear response to external forcing are vulnerable to the time scale separation due to increased response errors at fast scales. Here we develop the approximate linear response algorithm for slow variables in a two-scale dynamical system with explicit separation of slow and fast variables, which has improved numerical stability and reduced computational expense.

1. Introduction

Recently, Majda and the author [1–4] developed and tested a novel computational algorithm for predicting the mean response of nonlinear functions of states of a chaotic dynamical system to small change in external forcing via the fluctuation–dissipation theorem (FDT). This algorithm (called the short-time FDT (ST-FDT) algorithm in [2–4]) takes into account the fact that the dynamics of chaotic nonlinear forced-dissipative systems often reside on chaotic fractal attractors, where the classical quasi-Gaussian formula of the fluctuation–dissipation theorem often fails to produce satisfactory response prediction, especially in dynamical regimes with weak and moderate chaos and slower mixing. It has been discovered that the ST-FDT algorithm is an extremely precise response approximation for short response times, and can be blended with the classical quasi-Gaussian FDT algorithm (qG-FDT) for longer response times to alleviate negative effects of expanding Lyapunov directions. Additionally, in [1] the author developed a computationally inexpensive approximate method for ST-FDT using the reduced-rank tangent map. Majda and Wang [9] developed a comprehensive linear response framework in the case of non-autonomous dynamics with time-periodic forcing (which also applies for general non-autonomous dynamics).

However, in multiscale dynamical systems with time scale separation the ST-FDT method can be vulnerable to the presence of the fast variables, especially when the response is practically needed only for slow model variables (such as those in a climate system), due to increased response errors at fast scales. Moreover, it is often the case that there are only a few slow variables in the model and a large number of fast variables. Even if only the response of the slow variables is needed, in a straightforward implementation such as that in [3,2,4], the ST-FDT response operator has to be computed for all variables in the model, which can be computationally expensive or even practically impossible for models with large sets of fast variables.

In the work, we develop an approximate response algorithm based on averaged dynamics of multiscale systems [10,12,13]. The new method allows to compute the response operators directly at slow variables using existing FDT formulas, which improves numerical stability and reduces computational expense.

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2. Average response for two-scale systems perturbed at slow variables

Consider a two-scale system of Itô stochastic differential equations of the form

$$\begin{aligned}\frac{d\boldsymbol{x}}{dt} &= \boldsymbol{F}(\boldsymbol{x},\boldsymbol{y},t),\\ d\boldsymbol{y} &= \frac{1}{\varepsilon}\boldsymbol{G}(\boldsymbol{x},\boldsymbol{y},t,\varepsilon)dt + \frac{1}{\sqrt{\varepsilon}}\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{y},t,\varepsilon)d\boldsymbol{W}_t,\end{aligned} \tag{2.1}$$

where $\boldsymbol{x} = \boldsymbol{x}(t) \in \mathbb{R}^{N_x}$ is the set of slow variables, and $\boldsymbol{y} = \boldsymbol{y}(t) \in \mathbb{R}^{N_y}$ is the set of fast variables. This situation is common in geophysical science, where the time scale of different state variables of a weather/climate system can range between minutes and months (or even years), and fast variables are often driven by a random process. In (2.1) we use the following notations: $\boldsymbol{W}_t$ is the K-dimensional Wiener process, $\boldsymbol{F}$ and $\boldsymbol{G}$ are N_x and N_y vector-valued functions of $\boldsymbol{x}$, $\boldsymbol{y}$ and t, and $\boldsymbol{\sigma}$ is a $N_y \times K$ matrix-valued function of $\boldsymbol{x}$, $\boldsymbol{y}$ and t. For the purpose of this work, here we assume that there is a constant parameter $0 < \varepsilon \ll 1$ which sets the time scale separation between $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ into slow and fast variables, respectively, and we additionally assume that $\boldsymbol{G}$ and $\boldsymbol{\sigma}$ continuously depend on ε.

We make a general assumption that, given a time t_0 and a probability measure $\mathcal{P}_{t_0}$, the non-autonomous dynamical system in (2.1) transports it into $\mathcal{P}_{t_0+t}$, where t is the elapsed interval of time after t_0. Then, by ρ_{t_0+t} we denote the marginal measure of $\mathcal{P}_{t_0+t}$ for the set of slow variables $\boldsymbol{x}$, such that for any observable $A(\boldsymbol{x})$ its average value $\langle A\rangle(t_0+t)$ is given by

$$\langle A\rangle(t_0+t) = \rho_{t_0+t}(A) \equiv \int_{\mathbb{R}^{N_x}} A(\boldsymbol{x})d\rho_{t_0+t}(\boldsymbol{x}). \tag{2.2}$$

We say that the system in (2.1) is *perturbed at slow variables* when there is a small forcing $\boldsymbol{w}(\boldsymbol{x})\delta\boldsymbol{f}(t)$ applied at slow variables:

$$\begin{aligned}\frac{d\boldsymbol{x}}{dt} &= \boldsymbol{F}(\boldsymbol{x},\boldsymbol{y},t) + \boldsymbol{w}(\boldsymbol{x})\delta\boldsymbol{f}(t),\\ d\boldsymbol{y} &= \frac{1}{\varepsilon}\boldsymbol{G}(\boldsymbol{x},\boldsymbol{y},t,\varepsilon)dt + \frac{1}{\sqrt{\varepsilon}}\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{y},t,\varepsilon)d\boldsymbol{W}_t,\end{aligned} \tag{2.3}$$

where $\boldsymbol{w}(\boldsymbol{x})$ is an $N_x \times L$ matrix-valued function of $\boldsymbol{x}$, while $\delta\boldsymbol{f}(t)$ is a L vector-valued function of time t for some integer L. For the perturbed system in (2.3) we assume that $\mathcal{P}_{t_0}$ is transported into $\mathcal{P}^*_{t_0+t}$ with the corresponding marginal measure $\rho^*_{t_0+t}$ for slow variables. Finally, we define the *average response* of $A(\boldsymbol{x})$ to the small forcing in (2.3), starting at t_0, as

$$\delta\rho_{t_0+t}(A) = \rho^*_{t_0+t}(A) - \rho_{t_0+t}(A). \tag{2.4}$$

Our goal here is to compute a linearization of (2.4) with respect to the forcing $\boldsymbol{w}(\boldsymbol{x})\delta\boldsymbol{f}(t)$ in (2.3) under the assumption that both the forcing and the response are small, which is provided by FDT [1,3,2,4,8]. Observe that both the forcing $\boldsymbol{w}(\boldsymbol{x})\delta\boldsymbol{f}(t)$ and the response function $A(\boldsymbol{x})$ involve only slow variables $\boldsymbol{x}$. However, a straightforward application of the FDT linearization of the response to (2.4) will lead to the computation of the linear response operator for the complete set of model variables, that is, for both $\boldsymbol{x}$ and $\boldsymbol{y}$. This is undesirable for the following reasons: first, it can make the computation of the response expensive (especially if there are many fast variables, which is often the case); and, second, the stability of the ST-FDT response operator may suffer due to large Lyapunov exponents at fast variables. In what follows we develop the approximate FDT formulas under the assumption that the behavior of the slow variables in (2.1) approaches the limiting case of "infinitely fast" $\boldsymbol{y}$-variables.

3. Limiting dynamics for slow variables

We rescale the time in (2.1) and (2.3) as $t = \varepsilon\tilde{t}$. For the rescaled time $\tilde{t}$, the equation for fast variables $\boldsymbol{y}$ in both (2.1) and (2.3) becomes

$$d\boldsymbol{y} = \boldsymbol{G}(\boldsymbol{x},\boldsymbol{y},\varepsilon\tilde{t},\varepsilon)d\tilde{t} + \boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{y},\varepsilon\tilde{t},\varepsilon)d\boldsymbol{W}_{\tilde{t}}. \tag{3.1}$$

Following [10,12,13], in the vicinity of some $\boldsymbol{x}$ and t, we write the separate degenerate system for the fast variables in the limiting form as $\varepsilon \to 0$:

$$d\boldsymbol{z} = \boldsymbol{G}(\boldsymbol{x},\boldsymbol{z},t,0)d\tilde{t} + \boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{z},t,0)d\boldsymbol{W}_{\tilde{t}}, \tag{3.2}$$

where $\boldsymbol{x}$ and t are treated as constant parameters, and, therefore, $\boldsymbol{z}$ parametrically depends on $\boldsymbol{x}$ and t. Observe that the limiting system for the fast variables in (3.2) is autonomous, that is, it does not explicitly depend on $\tilde{t}$ (except for the Wiener process). Here we assume that (3.2) possesses the invariant ergodic probability measure $\mu_{\boldsymbol{x},t}$, which depends on $\boldsymbol{x}$ and t as parameters.

Now, following [10,12,13] we write the *averaged* perturbed and unperturbed systems for the slow variables $\boldsymbol{x}$ as

$$\frac{d\boldsymbol{x}}{dt} = \overline{\boldsymbol{F}}(\boldsymbol{x},t), \tag{3.3}$$

$$\frac{d\boldsymbol{x}}{dt} = \overline{\boldsymbol{F}}(\boldsymbol{x},t) + \boldsymbol{w}(\boldsymbol{x})\delta\boldsymbol{f}(t), \tag{3.4}$$