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Near-field imaging of biperiodic surfaces for elastic waves

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ABSTRACT

This paper is concerned with the direct and inverse scattering of elastic waves by biperiodic surfaces in three dimensions. The surface is assumed to be a small and smooth perturbation of a rigid plane. Given a time-harmonic plane incident wave, the direct problem is to determine the displacement field of the elastic wave for a given surface; the inverse problem is to reconstruct the surface from the measured displacement field. The direct problem is shown to have a unique weak solution by studying its variational formulation. Moreover, an analytic solution is deduced by using the transformed field expansion method and the convergence is established for the power series solution. A local uniqueness is proved for the inverse problem. An explicit reconstruction formula is obtained and implemented by using the fast Fourier transform. The error estimate is derived for the reconstructed surface function, and it provides an insight on the trade-off among resolution, accuracy, and stability of the solution for the inverse problem. Numerical results show that the method is effective to reconstruct biperiodic scattering surfaces with subwavelength resolution.

1. Introduction

The scattering problems for acoustic and electromagnetic waves have been undergoing extensive studies over the years [18]. Recently, the elastic wave scattering problems have received much attention from both engineering and mathematical communities due to their significant applications in diverse scientific areas [2–5,16,20–25,30]. The purpose of this paper is to study the inverse scattering problem for elastic wave scattering by biperiodic surfaces in three dimensions. It is indispensable to analyze the corresponding direct problem in order to serve this goal. Specifically, we consider a biperiodic scattering surface which is assumed to be a small and smooth perturbation of a plane. The space above the surface is filled with a linear, homogeneous, and isotropic elastic medium, while the space below the surface is elastically rigid. A time-harmonic elastic plane wave is incident on the surface from above. We consider the resonance regime where the wavelength of the incident wave is comparable with the period of the surface. Given the incident field, the direct scattering problem is to determine the displacement field of the total wave for the known surface; the inverse problem is to reconstruct the surface from the measured displacement field of the total wave at a horizontal plane over the surface. The well-posedness was studied for the direct scattering problem in [2,4,5,20,23,24] for the two-dimensional case and in [22] for the three-dimensional case. The inverse scattering problem was also investigated theoretically for its uniqueness in [1] for the two-dimensional case, and numerically by using nonlinear optimization in [21] and the factorization method in [29] for the two-dimensional problem.

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In this work, we propose an effective method for solving quantitatively the three-dimensional inverse elastic scattering problem and seek to achieve super resolved resolution. Compared with the acoustic or electromagnetic counterparts, the elastic scattering problems are more challenging due to the coexistence of compressional and shear waves that travel at different speeds. In view of this fact, we utilize the Helmholtz decomposition to split the displacement field of the elastic wave into a superposition of its compressional part and shear part by introducing two potential functions. Using the Rayleigh expansion, we derive the transparent boundary conditions for each potential function and recast the problem into a coupled boundary value problem. Based on the assumption that the surface is a small and smooth perturbation of a rigid plane, we apply the transformed field expansion to convert the three-dimensional boundary value problem into a successive sequence of two-point boundary value problems in the frequency domain. The method begins with the change of variable to flatten the curved surface into a planar surface; then it resorts to the power series expansion and the Fourier series expansion to find an analytic solution for the direct problem. Using the closed form of the analytic solution, we deduce simple expressions for the leading and linear terms of the power series solution. Dropping all higher order terms, we linearize the inverse problem and obtain an explicit and elegant identity which links the Fourier coefficients of the scattering surface and the measured displacement field. The scattering surface is then reconstructed from the truncated Fourier series expansion. The method requires only a single incident field and is efficiently implemented by the fast Fourier transform. Numerical examples show the method is effective and robust to reconstruct the scattering surfaces with subwavelength resolution. We refer to [17,31,38–40] for transformed field expansion and related boundary perturbation methods for solving various direct scattering problems.

Moreover, we provide theoretical analysis to validate the proposed method. Using a variational formulation, we establish the well-posedness and obtain an energy estimate of the solution for the direct scattering problem. Using a similar variational approach, we obtain the well-posedness of the solution for the recursive boundary value problems and prove the convergence of the power series solution. For the inverse problem, we show a local uniqueness result of the solution for a sufficiently small perturbation. We derive an error estimate for the reconstruction formula, which demonstrates a dependence of the reconstructed surface on all the physical parameters of the model problem and provides an insight on the trade-off among resolution, accuracy, and stability of the solution for the inverse problem. This paper is a non-trivial extension of our previous work on the two-dimensional elastic scattering problems [36,37] due to the obvious difference and increased challenge of the model problem. This work adds a significant contribution to our recent development of designing novel computational methods for solving a class of acoustic and electromagnetic inverse scattering problems [6,10–12,19,32–35]. We refer to [7–9,13–15,28] for other related inverse scattering problems for acoustic and electromagnetic waves.

The outline of the paper is as follows. In section 2 we introduce the model problem and the transparent boundary condition. Section 3 is devoted to the direct scattering problem where we prove the well-posedness of the solution by studying the variational formulation. In section 4 we present the transformed field expansion, derive the recursive boundary value problems, and prove the convergence of the power series expansion. The reconstruction formula, error estimate, and numerical examples are provided in section 5. We conclude the paper with comments and directions for future research in section 6.

2. Problem formulation

In this section we introduce a mathematical model for the elastic scattering by a biperiodic surface and derive a transparent boundary condition for the truncated problem.

2.1. Elastic wave equation

Let $\rho = (x, y) \in \mathbb{R}^2$ and $\boldsymbol{x} = (\rho, z) \in \mathbb{R}^3$. Let $\Lambda = (\Lambda_1, \Lambda_2)$, where $\Lambda_j > 0$ are constants. Denote a rectangular domain $R = (0, \Lambda_1) \times (0, \Lambda_2)$. Consider the part of a biperiodic surface in one periodic cell R:

$$\Gamma_f = \{\boldsymbol{x} \in \mathbb{R}^3 : z = f(\rho),\ \rho \in R\},$$

where $f \in C^k(R)$, $k \geq 2$ is a biperiodic function with period Λ. We assume that

$$f(\rho) = \varepsilon g(\rho), \tag{2.1}$$

where $\varepsilon > 0$ is a small constant and is called the surface deformation parameter, $g \in C^k(R)$ is a biperiodic function with period Λ and represents the normalized profile of f. Let

$$K = \max_{|s| \leq k} \sup_{\rho \in R} \left| D^s g(\rho) \right|, \tag{2.2}$$

which indicates the smoothness of the scattering surface and plays an important role in the subsequent convergence and error analysis. Denote

$$\Omega_f = \{\boldsymbol{x} \in \mathbb{R}^3 : z > f(\rho),\ \rho \in R\},$$

and