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Short note

Extension of Kleiser and Schumann's influence-matrix method for generalized velocity boundary conditions

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ABSTRACT

In 1980, Kleiser and Schumann introduced a novel influence-matrix method to treat the incompressibility and no-slip boundary conditions when solving the Navier–Stokes equations. They also outlined the related "tau" error correction technique which is essential for the high accuracy direct numerical simulation (DNS) of turbulent flows. However, their method is not valid for Robin type velocity boundary conditions (i.e., $B(\mathbf{u}) = \alpha\mathbf{u} + \beta\mathbf{u}' - \gamma = 0$). In this note, a new influence-matrix method is introduced where the boundary condition and "tau" correction are enforced in one step using an extended influence matrix. The new method is simple and easy to be implemented. It broadens the applicability of the Kleiser and Schumann method. Examples with the new method show excellent agreement with data in the literature and the velocity field is divergence free up to machine precision.

1. Introduction

Kleiser and Schumann [1] introduced a novel method to treat the incompressibility and boundary conditions when solving the Navier–Stokes equations for flows bounded by two parallel no-slip walls. Werne [2], based on the understanding of the original paper by Kleiser and Schumann [1] and other reproductions of this method (such as [3]), proposed a revised algorithm which treats the "tau" error on the "A"-problem level. In response, Kleiser et al. [4] pointed out that the original method in [1] has no error. In fact, both methods are correct and equally applicable. However, both methods are limited to channel flows bounded by no-slip walls.

This note contributes to the extension of the boundary conditions to the Robin type. The new method, as well as the original method, is based on the linearity of the coupled Helmholtz equation systems for pressure and wall normal velocity. However, the complications associated with the generalized boundary conditions make the original method not applicable. The new method will be described next and be demonstrated by an example.

2. Solution for the Helmholtz equations

It is assumed that the domain of the flow field is periodic in the streamwise (x) and spanwise (z) directions such that the incompressible Navier–Stokes equations can be Fourier transformed in both directions. In the wall normal (y)-direction, Chebyshev expansion is used. In this paper, u, v, and w will be used to denote, respectively, the velocity components in streamwise, wall normal, and spanwise directions. All quantities are normalized by wall shear velocity u_τ, the channel half width h, and the fluid viscosity ν, unless otherwise specified. Numerically integrating the Fourier transformed governing

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equation in time and after some manipulations (see for example [3]), the problem is reduced to a coupled system of 1D Helmholtz equations for wall normal velocity and pressure

$$\hat{p}'' - k^2\hat{p} = \hat{\nabla}\cdot\widetilde{\mathbf{R}} \tag{1}$$

$$\nu\hat{v}'' - \lambda\hat{v} - \hat{p}' = -R_y \tag{2}$$

with the boundary conditions

$$B_1(\hat{v}) = 0, \quad B_2(\hat{v}) = 0 \quad \text{at } y = -1 \text{ and } y = +1 \tag{3}$$

The primitives denote the derivatives with respect to y. The hat means the variables have been transformed into Fourier space. The right hand sides of both equations are terms resulting from the aforementioned operations. The two boundary conditions are derived from the slip/no-slip nature of the fluid and the divergence free requirement at the boundary. Without loss of generality, B_1 operator is designated as the former and B_2 operator for the later. In general, the boundary condition operators B_1 and B_2 in Eq. 3 are of Robin type, i.e., $B(\hat{v}) = \alpha\hat{v} + \beta\hat{v}' - \gamma$. Eqs. (1)–(3) forms the so-called A-problem.

The equations for $\hat{u}$ and $\hat{w}$ read

$$\nu\hat{u}'' - \lambda\hat{u} - ik_x\hat{p}' = -R_x \tag{4}$$

$$\nu\hat{w}'' - \lambda\hat{w} - ik_z\hat{p}' = -R_z \tag{5}$$

where k_x and k_z are wave numbers in streamwise and spanwise directions, respectively. The general boundary conditions of Robin type for $\hat{u}$ and $\hat{w}$ can be written as

$$\alpha_{u-}\hat{u} + \beta_{u-}\hat{u}' = \gamma_{u-}\delta_{k_x,k_z} \tag{6}$$

$$\alpha_{u+}\hat{u} + \beta_{u+}\hat{u}' = \gamma_{u+}\delta_{k_x,k_z} \tag{7}$$

$$\alpha_{w-}\hat{w} + \beta_{w-}\hat{w}' = \gamma_{w-}\delta_{k_x,k_z} \tag{8}$$

$$\alpha_{w+}\hat{w} + \beta_{w+}\hat{w}' = \gamma_{w+}\delta_{k_x,k_z} \tag{9}$$

Here the subscripts − and + are used to denote at the boundary $y = -1$ and $y = +1$, respectively. The α, β, and γ coefficients are assumed to be constant on each boundary. δ_{k_x,k_z} is a two-dimensional Dirac delta function.

Upon Chebyshev tau discretization, the discrete A-problem can be written as

$$p_m^{(2)} - k^2 p_m = r_m - Db_m, \quad m = 0,\ldots,N-2 \tag{10}$$

$$\nu v_m^{(2)} - \lambda v_m - p_m^{(1)} = -R_{y,m} + b_m, \quad m = 0,\ldots,N-2 \tag{11}$$

$$B_1 v_m(+-1) = 0 \tag{12}$$

$$B_2 v_m(+-1) = 0 \tag{13}$$

Here $b_m = 0$ when $m = 0, \ldots, N - 2$, while b_{N-1} and b_N are nonzero due to the tau errors, which have to be accounted for properly to satisfy the divergence free condition.

There are two separate subproblems which we shall introduce in the following sections. These two subproblems are distinguished by the coefficients of the general boundary conditions in Eqs. (6)–(9). For the first subproblem, the Helmholtz equations for $\hat{p}$ and $\hat{v}$ can be solved separately from $\hat{u}$ and $\hat{w}$. For the second subproblem, the four Helmholtz equations for $\hat{p}, \hat{u}, \hat{v}$, and $\hat{w}$ are coupled together and a more general technique has to be used. In Chebyshev collocation method, this complication of coupling has been dealt with properly by solving an extended A-problem (see for example [5]). The method used in this paper is similar with the exception of Chebyshev tau discretization of the Helmholtz equations.

2.1. Subproblem 1: $\hat{p}$–$\hat{v}$ decoupled from $\hat{u}$ and $\hat{w}$

Theorem 1. *The $\hat{p}$–$\hat{v}$ problem decouples from the $\hat{u}$ and $\hat{w}$ equations when the α and β coefficients in the boundary conditions for $\hat{u}$ and $\hat{w}$ are equal, i.e., $\alpha_{u-} = \alpha_{w-} = \alpha_-$, $\beta_{u-} = \beta_{w-} = \beta_-$, $\alpha_{u+} = \alpha_{w+} = \alpha_+$ and $\beta_{u+} = \beta_{w+} = \beta_+$. The boundary condition operator B_2 for the $\hat{p}$–$\hat{v}$ problem required by continuity are $\alpha_-\hat{v}' + \beta_-\hat{v}'' = 0$ and $\alpha_+\hat{v}' + \beta_+\hat{v}'' = 0$.*

Proof. To satisfy the continuity equation at the boundary $y = -1$, it is equivalent to require

$$\hat{\nabla}\cdot\hat{\mathbf{u}} = ik_x\hat{u} + ik_z\hat{w} + \hat{v}' = 0. \tag{14}$$

Taking the derivative of Eq. (14) with respect to y, one gets

$$ik_x\hat{u}' + ik_z\hat{w}' + \hat{v}'' = 0. \tag{15}$$

Multiplying Eqs. (14) and (15) by α_- and β_-, respectively and taking the sum give,

$$\alpha_-\hat{v}' + \beta_-\hat{v}'' = -[ik_x(\alpha_-\hat{u} + \beta_-\hat{u}') + ik_z(\alpha_-\hat{w} + \beta_-\hat{w}')] = -[ik_x\gamma_{u-}\delta_{k_x,k_z} + ik_z\gamma_{w-}\delta_{k_x,k_z}] = 0$$