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Solving eigenvalue problems on curved surfaces using the Closest Point Method

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ABSTRACT

Eigenvalue problems are fundamental to mathematics and science. We present a simple algorithm for determining eigenvalues and eigenfunctions of the Laplace–Beltrami operator on rather general curved surfaces. Our algorithm, which is based on the Closest Point Method, relies on an embedding of the surface in a higher-dimensional space, where standard Cartesian finite difference and interpolation schemes can be easily applied. We show that there is a one-to-one correspondence between a problem defined in the embedding space and the original surface problem. For open surfaces, we present a simple way to impose Dirichlet and Neumann boundary conditions while maintaining second-order accuracy. Convergence studies and a series of examples demonstrate the effectiveness and generality of our approach.

1. Introduction

The study of eigenvalues and eigenfunctions of the Laplacian operator has long been a subject of interest in mathematics, physics, engineering, computer science and other disciplines. Of considerable importance is the case where the underlying domain is a curved surface, S, in which case the problem becomes one of finding eigenvalues and eigenfunctions of the corresponding Laplace–Beltrami operator

$$-\nabla_S \cdot \nabla_S u = \lambda u, \tag{1}$$

or, more generally, the elliptic operator

$$-\nabla_S \cdot (a(x)\nabla_S u) = \lambda u.$$

The Laplace–Beltrami eigenvalue problem has played a prominent role in recent years in data analysis. For example, in [1], eigenvalues of the Laplace–Beltrami operator were used to extract "fingerprints" which characterize surfaces and solid objects. In [2,3], Laplace–Beltrami eigenvalues and eigenfunctions were used for dimensionality reduction and data representation. Other application areas include smoothing of surfaces [4] and the segmentation and registration of shape [5].

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Analytical solutions to the Laplace–Beltrami eigenvalue problem are rarely available, so it is crucial to be able to numerically approximate them in an accurate and efficient manner. Partial differential equations on surfaces, including eigenvalue problems, have traditionally been approximated using either (a) discretizations based on a parameterization of the surface [6], (b) finite element discretizations on a triangulation of the surface [7], or (c) embedding techniques which solve some embedding PDE in a small region near the surface [8] (see also the related works [9–15]).

Parameterization methods (a) are often effective for simple surfaces [6], but for more complicated geometries have the deficiency of introducing distortions and singularities into the method through the parameterization [16]. Approaches based on the finite element method can be deceptively difficult to implement; as described in [7], "even though this method seems to be very simple, it is quite tricky to implement". Embedding methods (c) have gained a considerable following because they permit PDEs on surfaces to be solving using standard finite differences.

This paper proposes a simple and effective embedding method for the Laplace–Beltrami eigenvalue problem based on the Closest Point Method. The Closest Point Method is a recent embedding method that has been used to compute the numerical solution to a variety of partial differential equations [17–20], including in-surface heat flow, reaction-diffusion equations, and higher-order motions involving biharmonic and "surface diffusion" terms. Unlike traditional embedding methods, which are built around level set representatives of the surface, the Closest Point Method is built around a closest point representation of the surface. This allows for general smooth surfaces with boundaries and does not require the surface to have an inside/outside [17]. In addition, the method does not introduce artificial boundary conditions at the edge of the computation band. Such artificial boundary conditions typically lead to low-order accuracy [12].

Here we apply the Closest Point Method to the problem of determining the eigenvalues and eigenmodes of the Laplace–Beltrami operator on a surface. We begin by demonstrating that, for closed surfaces, there is a one-to-one correspondence between the eigenvalues of the embedding problem and the original surface problem. Later, we consider open surfaces and present simple techniques for achieving high-order accurate approximations to Dirichlet and homogeneous Neumann boundary conditions. Our proposed method retains the usual advantages of the Closest Point Method, namely generality with respect to the surface, high-order accuracy and simplicity of implementation.

The paper unfolds as follows. Section 2 provides key background on the Closest Point Method. Section 3 proposes an embedding problem used to solve the Laplace–Beltrami eigenvalue problem and explains why a similar embedding problem leads to spurious eigenvalues. Section 4 provides discretization details. In Section 5, a second-order discretization of boundary conditions is described for open surfaces. Section 6 validates the method with a number of convergence studies and examples on complex shapes. Finally, Section 7 gives a summary and conclusions.

2. The Closest Point Method

We now review the ideas behind the Closest Point Method [17] which are relevant to the problem of finding Laplace–Beltrami eigenvalues and eigenfunctions.

The representation of the underlying surface is fundamental to any numerical method for PDEs on surfaces. The Closest Point Method relies on a closest point representation of the underlying surface.

Definition 1 (*Closest point function*). Given a surface $\mathcal{S}$, $\mathrm{cp}(\boldsymbol{x})$ refers to a (possibly non-unique) point belonging to $\mathcal{S}$ which is closest to x.

The closest point function, defined in a neighborhood of a surface, gives a representation of the surface. This closest point representation allows for general surfaces with boundaries and does not require the surface to have an inside/outside. The surface can be of any codimension [17], or even of mixed codimension [20].

The goal of the Closest Point Method is to replace a surface PDE by a related PDE in the embedding space which can be solved using finite difference, finite element or other standard methods. In the case of the Laplace-Beltrami eigenvalue problem, this approach relies on the following result, which states that the Laplace–Beltrami operator $\Delta_{\mathcal{S}}$ may be replaced by the standard Laplacian Δ in the embedding space $\mathbb{R}^d$ under certain conditions.

Theorem 1. *Let $\mathcal{S}$ be a smooth closed surface in $\mathbb{R}^d$ and $u : \mathcal{S} \to \mathbb{R}$ be a smooth function. Assume the closest point function cp(x) is defined in a neighborhood $\Omega \subset \mathbb{R}^d$ of $\mathcal{S}$. Then*

$$\Delta_{\mathcal{S}} u(\boldsymbol{x}) = \Delta(u(\mathrm{cp}(\boldsymbol{x}))) \quad \text{for } \boldsymbol{x} \in \mathcal{S}. \tag{2}$$

Note that the right-hand side is well-defined because u(cp(·)) can be evaluated at points both on and off the surface.
This result follows from the principles in [17].

Because the function $u(\mathrm{cp}(\boldsymbol{x}))$, known as the closest point extension of u, is used throughout this paper, we make the following definition.

Definition 2 (*Closest point extension*). Let $\mathcal{S}$ be a smooth surface in $\mathbb{R}^d$. The *closest point extension* of a function $u : \mathcal{S} \to \mathbb{R}$ to a neighborhood Ω of $\mathcal{S}$ is the function $v : \Omega \to \mathbb{R}$ defined by

$$v(\boldsymbol{x}) = u(\mathrm{cp}(\boldsymbol{x})). \tag{3}$$