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A finite element dynamical nonlinear subscale approximation for the low Mach number flow equations

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ABSTRACT

In this work we propose a variational multiscale finite element approximation of thermally coupled low speed flows. The physical model is described by the low Mach number equations, which are obtained as a limit of the compressible Navier–Stokes equations in the small Mach number regime. In contrast to the commonly used Boussinesq approximation, this model permits to take volumetric deformation into account. Although the former is more general than the latter, both systems have similar mathematical structure and their numerical approximation can suffer from the same type of instabilities.

We propose a stabilized finite element approximation based on the variational multiscale method, in which a decomposition of the approximating space into a coarse scale resolvable part and a fine scale subgrid part is performed. Modeling the subscale and taking its effect on the coarse scale problem into account results in a stable formulation. The quality of the final approximation (accuracy, efficiency) depends on the particular model.

The distinctive features of our approach are to consider the subscales as transient and to keep the scale splitting in all the nonlinear terms. The first ingredient permits to obtain an improved time discretization scheme (higher accuracy, better stability, no restrictions on the time step size). The second ingredient permits to prove global conservation properties. It also allows us to approach the problem of dealing with thermal turbulence from a strictly numerical point of view.

Numerical tests show that nonlinear and dynamic subscales give more accurate solutions than classical stabilized methods.

1. Introduction

The general description of a fluid flow involves the solution of the compressible Navier–Stokes equations. It is widely accepted that these equations provide an accurate description of any problem in fluid mechanics which may present many different nonlinear physical mechanisms. Depending on the physics of the problem under consideration, different simplified models describing some of these mechanisms can be derived from the compressible Navier–Stokes equations.

Our application is directed to low speed strongly thermally coupled flows which are described by the compressible Navier–Stokes equations in the low-Mach number limit. This limit is derived by an asymptotic expansion of the problem variables as power series of the small parameter $\gamma \mathrm{Ma}^2 \ll 1$, where γ denotes the specific heat ratio and Ma the Mach number of the problem. For details of this asymptotic expansion procedure, see [27,29,33]. As a particular result of this process, the total pressure is split into two parts, the thermodynamic part $p^{th}(t)$ which is uniform in space, and the hydrodynamic part

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$p(\boldsymbol{x}, t)$ which is several orders of magnitude smaller than p^{th} and is therefore omitted in the state and energy equations. This leads to a removal of the acoustic modes but large variations of density due to temperature variations are allowed. This system of equations is commonly used to describe problems of combustion in the form of deflagrations (i.e., flames at low speed).

Despite this important difference in the treatment of the incompressibility, the low Mach number equations present the same mathematical structure as the incompressible Navier–Stokes equations, in the sense that the mechanical pressure is determined from the mass conservation constraint. Consequently the same type of numerical instabilities can be found, namely the problem of compatibility conditions between the velocity and pressure finite element spaces, and the instabilities due to convection dominated flows. These instabilities can be avoided by the use of stabilization techniques. A Galerkin finite element method can be used with mixed LBB stable elements, avoiding stabilization techniques when convection is not dominant [18,30]. Stabilized finite element methods (FEM) have been initially developed for the Stokes [22] and for the convection diffusion reaction (CDR) problems [6]. Later they have been extended to incompressible Navier–Stokes equations [7,26] , and for the low Mach approximation [32] but the nonlinearity of the problem was not considered in their design. These extensions were essentially the application to nonlinear transient problems of a technique developed for linear steady ones.

The design of stabilization techniques considering the transient nonlinear nature of the problems began with the introduction of dynamic nonlinear subscales in [8,12]. Developed in the context of the variational multiscale (VMS) concept introduced by Hughes [21] , the idea is to consider the subgrid scale time dependent and to consider its effect on all the nonlinear terms, resulting in extra terms in the final discrete scheme. Important improvements in the discrete formulation of the incompressible Navier–Stokes problem have been observed. From a theoretical point of view, the use of transient subgrid scales explains how the stabilization parameter should depend on the time step size and makes space and time discretizations commutative. The tracking of the subscales along the nonlinear process provides global momentum conservation for incompressible flows. From a practical point of view, the use of time dependent nonlinear subscales results in a more robust and more accurate method (an unusual combination) as shown by numerical experiments [8,12].

These developments also opened the door to the use of numerical techniques to cope with the potential instabilities and to model turbulence at the same time, as pointed out in [8,12]. This is a natural step as turbulence is originated by the presence of the nonlinear convective term, as it is well known. The idea of a large eddy simulation (LES) approach to turbulence modeling using only numerical ingredients actually goes back at least to [5], and the possibility to use the VMS framework for that purpose to [8]. It was fully developed for incompressible flows in [3] and for low Mach number flows recently in [15], where quantitative comparisons against direct numerical simulations are presented. It is important to point out, however, that not all the terms arising from the nonlinear scale splitting are considered in these works. Apart from these results, a careful analysis of the dissipative structure of the variational multiscale method with nonlinear time dependent subscales was presented in [17,34], showing the physical interpretation of the method. This analysis was extended to thermally coupled flows using the Boussinesq approximation in [10].

In this article we consider time dependent subscales and their effect in *all* the nonlinear terms in the low Mach number flow equations. It is shown that the method does not only provide the necessary stabilization of the formulation but also enables to obtain more accurate solutions than the classical linear approach for an equivalent mesh as it happened for incompressible flows. It is also shown that global conservation properties for mass, momentum and energy are obtained from the final discrete scheme.

The paper is organized as follows. In Section 2, the Low Mach number equations and their variational formulation are given. Afterwards the VMS formulation through dynamic scale splitting is derived in Section 3. It is shown in Section 4 that this formulation provides global mass, momentum and energy conservation when using equal interpolation spaces for the velocity, pressure and temperature equations. Time integration schemes are discussed in Section 5. The treatment of the nonlinear terms is described in detail in Section 6. The formulation is tested for both stationary and dynamic problems in Section 7. Conclusions are drawn in Section 8.

2. The low Mach number equations

2.1. Initial and boundary value problem

Let $\Omega \subset \mathbb{R}^d$, with $d = 2,3$ be the computational domain in which the flow takes place during the time interval $[0, t_{\text{end}}]$, and let $\partial\Omega$ be its boundary. The initial and boundary value problem to be considered consists of finding a velocity field $\boldsymbol{u}$, a hydrodynamic pressure field p, a temperature field T, and the thermodynamic pressure p^{th} such that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \text{ in } \Omega, \quad t \in (0, t_{\text{end}}) \tag{1}$$

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nabla \cdot (2\mu \boldsymbol{\varepsilon}'(\boldsymbol{u})) + \nabla p = \rho \boldsymbol{g} \text{ in } \Omega, \quad t \in (0, t_{\text{end}}) \tag{2}$$

$$\rho c_p \frac{\partial T}{\partial t} + \rho c_p \boldsymbol{u} \cdot \nabla T - \nabla \cdot (k \nabla T) - \alpha T \frac{dp^{th}}{dt} = Q \text{ in } \Omega, \quad t \in (0, t_{\text{end}}) \tag{3}$$