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Sinc collocation approximation of non-smooth solution of a nonlinear weakly singular Volterra integral equation

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ABSTRACT

A numerical method based on sinc collocation approximation for a class of nonlinear weakly singular Volterra integral equations of a second kind with non-smooth solution is given. The numerical method given here combines a sinc collocation method with an explicit iterative process that involves solving a nonlinear system of equations. We provide an error analysis for the method. It is shown that the approximate solution converges to the exact solution at the rate of $\sqrt{M}\exp(-c\sqrt{M})$, where M is the number of collocation points and c is some positive constant. Some numerical results for several test functions are given to confirm the accuracy and the ease of implementation of the method.

1. Introduction

Many physical, chemical, and biological problems are modeled as nonlinear Volterra integral equations, such as reaction–diffusion problems, crystal growth, models describing the propagation of a flame (see e.g. [23,10] and especially [14] for many physical and engineering applications), mathematical models describing the behavior of viscoelastic materials in mechanics, superfluidity problems, and some newer applications on the study of soft tissues like mitral valves of the aorta in human heart (see [9] and the references therein).

This work is concerned with study of the numerical analysis of a class of nonlinear Volterra integral equation of a second kind which has a weakly singular kernel of the form

$$u(x) = f(x) + \int_a^x \frac{K(x,t)}{(x-t)^\alpha} u^p(t)\,dt, \tag{1}$$

where $a \leqslant x, t \leqslant b$, $p > 1$, and $0 < \alpha < 1$. Eq. (1) can arise in connection with some heat conduction problems with various class of mixed-type boundary conditions. For example, Lighthill [15] was among the pioneers to derive an integral equation that can be transformed into the above equation which describes the temperature distribution of the surface of a projectile moving through a laminar layer when $f(x) = 1,\ K(x,t) = (-\sqrt{3}/\pi)t^{1/3},\ \alpha = 2/3,\ p = 4$ and $a = 0$, $b = 1$. Even for analytic functions $f(x)$ and $K(x,t)$, it is well known (e.g. see [5], and [1]) that derivative of the solution of the above equation, $u'(x)$ is singular at the left edge point of the interval of integration, $[a,x]$, and this is expected to cause a loss in global convergence of a collocation method. In the case of Eq. (1), $u'(x)$ behaves as $(x-a)^{-\alpha}$ as $x \to a^+$.

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Numerical approximations methods such as quadrature rules, finite differences, finite elements, and so on are generally use polynomials as basis functions to obtain approximate solutions that are sufficiently accurate in region where the function to be approximated is smooth (see e.g., [8,13]). However, such methods fail significantly in a neighborhood of singularities of the function. On the other hand, the numerical approximations obtained by using Whittaker's cardinal function yield much better results than those obtained by methods using polynomials in the case when singularities are present at an endpoint of the interval. These methods, however, may or may not yield better results in the absence of singularities. For a comprehensive study of numerical methods for Volterra integral equations we refer to [16,4,5], and the references therein, for single exponential sinc approximation methods to [17,26], and [27], for double exponential sinc transformation methods to [21,28] and [29].

In the present paper we develop a sinc collocation method for the nonlinear integral Eq. (1) that is based on the work of [24] for linear integral equation. [24] points out that "the extension of the method to nonlinear integral equations seems to be a more challenging task at this point." To our knowledge, no such extension is extant in the literature for the method given in [24]. Recently, [21] modified the method of [24] using a double exponential transformation for the linear case again and most recently [22] extending the method [24] to Fredholm case. In [24] the error analysis is based on an ambiguous limitation such as for all "M in a practical range," where M is the number of collocation points. In this paper, we set up the equations that gives the approximate solution for the integral equation in such a way that avoids such limitation. This is a sharp contrast between our approach and those in [24] and [21]. This paper is organized as follows. In Section 2 we present some definitions and preliminary results on sinc collocation method of single exponential function. Section 3 is devoted to a detailed derivation of our numerical algorithm, convergence and error analysis. Section 4 contains some numerical examples illustrating the applications of method described here that considers the rule of number of collocations. We end the paper with some closing remarks and conclusions.

2. Some preliminary results using sinc functions

In this section, we state some basic results about sinc function approximation. These important properties will enable us to solve the nonlinear singular Volterra integral equation. The basic sinc function is defined as

$$\operatorname{sinc}(x)=\begin{cases}\frac{\sin\pi x}{\pi x}, & x\neq 0,\\ 1, & x=0.\end{cases} \tag{2}$$

Let j be an integer and h be a positive number. We define the jth translate of sinc function by

$$S(j,h)(x)\equiv \operatorname{sinc}(x/h-j) \tag{3}$$

for step size h, evaluated at x. Given a function f defined and bounded for all x in $(-\infty,\infty)$, the *Whittaker's cardinal function* of f is defined by

$$C(f,h)(x)=\sum_{j=-\infty}^{\infty} f(jh)S(j,h)(x). \tag{4}$$

Now, we want to extend the approximations on $\mathbb{R}$ to the finite interval (a,b). Since the integral equation is defined over a finite interval, and the sinc function maps $\mathbb{R}$ onto a finite interval, we need some transformation $\phi(x)$ that maps a finite interval (a,b) onto $\mathbb{R}$. Let

$$\phi(z)=\log\left(\frac{z-a}{b-z}\right), \tag{5}$$

be a conformal map which carries the eye-shaped complex domain

$$D=\left\{z:\left|\arg\left(\frac{z-a}{b-z}\right)\right|<d<\pi\right\} \tag{6}$$

onto the open infinite strip

$$D_d=\{z\in\mathbb{C}:|\mathrm{Im}(z)|<d<\pi\}. \tag{7}$$

Note that at $x = kh$ with k an integer, the translate of sinc reduces to the Kronecher delta, i.e., $S(j,h)(kh) = \operatorname{sinc}(k-j) = \delta_{kj}$. We define the basis functions on (a,b) by

$$S(j,h)(\phi(x))=\operatorname{sinc}(\phi(x)/h-j). \tag{8}$$

Setting

$$\phi(x_k)=\log\left(\frac{x_k-a}{b-x_k}\right)=kh \tag{9}$$

we get

$$x_k=\phi^{-1}(kh)=\frac{a+be^{kh}}{1+e^{kh}}. \tag{10}$$