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Absorbing boundary conditions for scalar waves in anisotropic media. Part 2: Time-dependent modeling

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ABSTRACT

With the ultimate goal of devising effective absorbing boundary conditions (ABCs) for general anisotropic media, we investigate the well-posedness and accuracy aspects of local ABCs designed for the transient modeling of the scalar anisotropic wave equation. The ABC analyzed in this paper is the perfectly matched discrete layers (PMDL), a simple variant of perfectly matched layers (PML) that is also equivalent to rational approximation based ABCs. Specifically, we derive the necessary and sufficient condition for the well-posedness of the initial boundary value problem (IBVP) obtained by coupling an interior and a PMDL ABC. The derivation of the reflection coefficient presented in a companion paper (S. Savadatti, M.N. Guddati, J. Comput. Phys., 2010, doi:10.1016/j.jcp.2010.05.018) has shown that PMDL can correctly identify and accurately absorb outgoing waves with opposing signs of group and phase velocities provided the PMDL layer lengths satisfy a certain bound. Utilizing the well-posedness theory developed by Kreiss for general hyperbolic IBVPs, and the well-posedness conditions for ABCs derived by Trefethen and Halpern for isotropic acoustics, we show that this bound on layer lengths also ensures well-posedness. The time discretized form of PMDL is also shown to be theoretically stable and some instability related to finite precision arithmetic is discussed.

1. Introduction

Many wave propagation problems defined on physically unbounded domains can be divided into two regions; the interior and the exterior, with the interface between them termed the computational boundary. The interior is a small bounded region where the solution to the governing equations is sought while the exterior is the rest of the unbounded domain whose effect on the interior is required *only* at the computational boundary. The computational boundary is a boundary introduced solely for computational purposes and should be distinguished from physical boundaries. Since the solution to the governing equations is not required in the exterior, the computational domain can be restricted to just the interior by specifying appropriate absorbing boundary conditions (ABCs) at the computational boundary.

ABCs are thus used to replace a 'physical' model by an equivalent 'computational' model. The physical model consists of the interior and exterior governing equations, along with initial conditions (ICs) and physical boundary conditions (BCs) defined on the physical domain (interior + exterior). The computational model consists of the interior governing equations, ICs, physical BCs and ABCs defined on the computational domain (interior + computational boundary). Both these systems are initial boundary value problems (IBVPs) and will henceforth be referred to as 'physical IBVP' and 'computational IBVP', respectively. ABCs can thus be viewed as additional constraints on the physical IBVP that limit (or expand) the space of exist-

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ing solutions. If the constraints are too restrictive, all valid solutions of the physical IBVP might be excluded rendering the computational IBVP unsolvable. If they are too lax, spurious, unphysical solutions might be admitted rendering the computational IBVP inaccurate. Hence, it is essential to ensure that the ABCs used are 'appropriate'; appropriateness here being determined through the criteria of *well-posedness* and *accuracy*. Roughly speaking, well-posedness refers to the existence of a unique solution that is bounded in some way by the initial and boundary data of the computational IBVP, while accuracy refers to the close resemblance of this unique solution to the exact solution. In addition to these two, a third criterion, namely that of computational *efficiency*, is many a times required by large scale simulations [1,2].

Exact ABCs are well-posed and accurate by default, but their availability is restricted to simple exteriors with regular computational boundaries. Approximate ABCs provide acceptable accuracy and are available for more complicated problems, but their well-posedness is not guaranteed. For large scale simulations, however, exact ABCs are prohibitively expensive; this necessitates the use of approximate ABCs. Even amongst approximate ABCs, those containing nonlocal spatial and temporal operators (global ABCs) are unsuitable for large scale problems and hence local ABCs are preferred [1,2]. The most popular local ABCs currently available are rational ABCs and perfectly matched layers (PMLs) [3]. Rational ABCs approximate the exact stiffness of an exterior (or associated dispersion relation) with rational functions of varying orders; Lindman [4], Engquist and Majda [5,6], Bayliss and Turkel [7] and Higdon [8] were their early developers followed by many others [2]. Initial numerical implementations of rational ABCs were restricted to low orders but later auxiliary variable formulations provided practical high order rational ABCs [9]. The other popular local ABC, the PML, is a 'special' absorbing medium that uses complex coordinate stretching to dampen out (or decay) propagating waves without creating artificial reflections at the computational boundary. First introduced by Bérenger [11] and closely followed by the complex coordinate stretching viewpoint provided by Chew et al. [12–14], PMLs are now available in split and unsplit forms with variations like the conformal PML [15], complex frequency shifted PML (CFS-PML) [16], convolutional PML (CPML) [17] and multiaxial PML (M-PML) [18]. Currently, both rational ABCs and PMLs are available for a wide variety of governing equations that include, among many others, Maxwell's, linearized Euler's and elastodynamic equations.

Rational ABCs tend to be more accurate than PML because the effect of the rational ABC parameters on solution accuracy is better understood (and hence more easily handled). On the other hand, ABCs based on PML have proven to be more versatile by being easily extendible to complicated exteriors [3]. The term complicated here implies material complications like heterogeneities and/or anisotropy and geometrical complications like corners and conformal boundaries. While both local ABCs satisfy the criteria of accuracy and efficiency, neither of them is assured to be well-posed *per se*. The development of both these ABCs is fraught with examples of seemingly reasonable formulations that have been found to lack well-posedness in one sense or another e.g. see [19,20]. In fact, proving well-posedness (or stability) of newly formulated ABCs is now *de rigueur* e.g. see [5,6,8,21–33]. Studies focusing solely on well-posedness issues have been rare with some accessible papers being by Higdon [34], Trefethen et al. [35], Bécache et al. [26] and Appelö et al. [36]. The references within these papers can be used to get a more comprehensive review of previous works. While the mathematical well-posedness theories are well developed today and most of their physical implications have been understood [34], their application to specific governing equations is not always straightforward; especially for complicated media.

One of the challenges to devising well-posed local ABCs for complicated (anisotropic and/or inhomogeneous) media comes from the well-posedness criterion imposed on propagating waves. Well-posedness requires that an ABC should not admit propagating modes travelling into the interior (incoming modes) in the absence of outgoing modes and sources on the boundary [34]. This makes physical sense in as much as an ABC should not allow spontaneous emission of energy into the interior without interior or boundary excitation [35]. While propagating waves are distinguished into incoming and outgoing waves depending on their group velocity, rational ABCs and PML have both been traditionally formulated to absorb waves depending on their phase velocities. This dependence on phase velocities (instead of group velocities) does not affect simple media where the phase and group velocities are always of the same sign (e.g. homogeneous isotropic media) and hence ABC formulations for simple media have turned out to be well-posed. In fact, it has been shown that a condition necessary for stability of PML (a concept related to well-posedness in the sense of providing bounds for solutions) is the absence of wave modes with phase and group velocities of differing signs [26]. Recognizing the fact that many anisotropic and/or inhomogeneous media admit such wave modes, much recent research has been focused on developing techniques that result in well-posed (or stable) local ABCs for such media. A scalar anisotropic medium whose principal material axis is tilted with respect to the coordinate axis is a simple example of a medium that allows wave modes with differing phase and group velocity signs (see Sections 2.2 and 2.3). A similar challenge, arising from the existence of wave modes with inconsistent phase and group velocity signs, has already been recognized in the cases of anisotropic electromagnetism, advective acoustics and anisotropic elastodynamics, e.g. [24–33]. Most of these studies approach the well-posedness issues from a PML viewpoint and, for the particular case of advective acoustics, all of them specify linear space–time transformations that nullify the inconsistencies in phase and group velocity signs. Moreover, many of these studies model the problem as an initial value problem (IVP) and deal with the continuous form of the ABC prior to discretization. However, actual implementations of these ABCs are discretized IBVPs and it is not entirely clear how their behavior can be inferred from the continuous IVP results.

In this paper, we consider well-posedness from a rational ABC point of view and hence deal with IBVPs. Moreover, the ABC chosen for this purpose is the perfectly matched *discrete* layer (PMDL) [37–39] that is inherently discrete. We provide a necessary and sufficient criterion for well-posedness of PMDL, when it is used as an ABC for the scalar anisotropic wave equation. In fact, this criterion is also sufficient for accuracy and is the same as the accuracy condition developed for the time