Journal of Computational Physics 230 (2011) 7519–7532

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Stable Robin solid wall boundary conditions for the Navier–Stokes equations

Jens Berg [a,*], Jan Nordström [b]

[a] Uppsala University, Department of Information Technology, SE-751 05 Uppsala, Sweden
[b] Linköping University, Department of Mathematics, SE-581 83 Linköping, Sweden

ARTICLE INFO

Article history:
Received 15 April 2011
Received in revised form 21 June 2011
Accepted 24 June 2011
Available online 5 July 2011

Keywords:
Navier–Stokes
Robin boundary conditions
Well-posedness
Stability
High order accuracy
Summation-By-Parts

ABSTRACT

In this paper we prove stability of Robin solid wall boundary conditions for the compressible Navier–Stokes equations. Applications include the no-slip boundary conditions with prescribed temperature or temperature gradient and the first order slip-flow boundary conditions. The formulation is uniform and the transitions between different boundary conditions are done by a change of parameters. We give different sharp energy estimates depending on the choice of parameters.

The discretization is done using finite differences on Summation-By-Parts form with weak boundary conditions using the Simultaneous Approximation Term. We verify convergence by the method of manufactured solutions and show computations of flows ranging from no-slip to almost full slip.

1. Introduction

There has recently been a development of stable boundary [1,2] and interface [3] conditions of a specific form for the compressible Navier–Stokes equations. This paper extends the result in [2] to more general solid wall boundary conditions and includes sharp energy estimates. While [2] deals only with the no-slip boundary conditions, we will provide a uniform formulation which includes the no-slip boundary conditions with prescribed temperature or temperature gradient and slip-flow boundary conditions or any combination thereof.

The tools that we will use to obtain a uniform formulation together with proof of stability are finite difference approximations on Summation-By-Parts (SBP) form together with the Simultaneous Approximation term. This method has the benefit of being stable by construction for any linear well-posed Cauchy problem [4,5] and the robustness has been shown in a wide range of applications [5–9].

The first derivative is approximated by $u_x \approx Dv = P^{-1}Qv$, where v is the discrete grid function, D is the differentiation matrix, $P = P^T > 0$ defines a norm by $\|v\|^2 = v^TPv$ and Q has the SBP property $Q + Q^T = B = [-1, 0, \ldots, 0, 1]^T$, see [10,11] for details about these operators.

There exist operators accurate of order 2, 4, 6 and 8 and the stability analysis does not depend on the order of accuracy of the operators. We will pose our equations on conservative form and hence we do not need an operator approximating the second derivative. Operators approximating the second derivative with constant coefficients are derived in [11] and have recently been developed for variable coefficients problems in [12].

* Corresponding author. Address: Division of Scientific Computing, Department of Information Technology, Uppsala University, Box 337, SE-751 05 Uppsala, Sweden. Tel.: +46 18 471 6253; fax: +46 18 523049/511925.

E-mail address: jens.berg@it.uu.se (J. Berg).

0021-9991/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.jcp.2011.06.027

The boundary conditions will be imposed weakly using the Simultaneous Approximation Term (SAT). The SAT term is added to the right-hand-side of the discretized equations as a penalty term which forces the equation towards the boundary conditions. Together the SBP and SAT technique provide a tool for creating stable approximations for well-posed initial-boundary value problems. The relation between weak and strong boundary conditions in terms of accuracy is discussed in [13].

2. The Navier–Stokes equations

2.1. Continuous case

We consider the two-dimensional Navier–Stokes equations on conservative form

$$q_t + F_x + G_y = 0, \tag{1}$$

where

$$F = F^I - \varepsilon F^V, \quad G = G^I - \varepsilon G^V. \tag{2}$$

The superscript I denotes the inviscid part of the fluxes and V the viscous part. The components of the solution vector are $q = [\rho, \rho u, \rho v, e]^T$ which are the density, x- and y-directional momentum, respectively and energy. The components of the fluxes are given by

$$\begin{aligned} F^I &= [\rho u, p + \rho u^2, \rho u v, u(p+e)]^T, \\ G^I &= [\rho v, \rho u v, p + \rho v^2, v(p+e)]^T, \\ F^V &= [0, \tau_{xx}, \tau_{xy}, u\tau_{xx} + v\tau_{xy} - Q_x]^T, \\ G^V &= [0, \tau_{xy}, \tau_{yy}, u\tau_{yx} + v\tau_{yy} - Q_y]^T, \end{aligned} \tag{3}$$

where p is the pressure, Pr the Prandtl number, γ the ratio of specific heat and $Q = -\kappa T$ is the thermal conductivity times the temperature according to Fourier's law. The stress tensors are given by

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \quad \tau_{yy} = 2\mu \frac{\partial v}{\partial y} + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \quad \tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{4}$$

where μ and λ are the dynamic and second viscosity, respectively.

All the equations above have been non-dimensionalized as

$$u = \frac{u^*}{c^*_\infty}, \quad v = \frac{v^*}{c^*_\infty}, \quad \rho = \frac{\rho^*}{\rho^*_\infty}, \quad T = \frac{T^*}{T^*_\infty}, \quad p = \frac{p^*}{\rho^*_\infty (c^*_\infty)^2}, \quad e = \frac{e^*}{\rho^*_\infty (c^*_\infty)^2}, \quad \lambda = \frac{\lambda^*}{\mu^*_\infty}, \quad \mu = \frac{\mu^*}{\mu^*_\infty}, \tag{5}$$

where the $*$-superscript denotes a dimensional variable and the ∞-subscript the freestream value. In (2) we have $\varepsilon = \frac{Ma}{Re}$ where Ma is the Mach-number and $Re = \frac{\rho^*_\infty u^*_\infty L^*_\infty}{\mu^*_{\inf ty}}$ is the Reynolds-number with L^*_∞ being a characteristic length scale.

The equations as stated in (1) is a highly non-linear system of equations. The well-posedness and stability conditions that will be derived in this paper will be based on a linear symmetric formulation.

We freeze the coefficients at some constant state $\bar{w} = [\bar{\rho}, \bar{u}, \bar{v}, \bar{p}]^T$ and linearize as $\tilde{w} = \bar{w} + w'$ where $w' = [\rho, u, v, p]^T$ is a perturbation around the constant state $\bar{w}$. This yields an equation with constant matrix coefficients. Next we transform to primitive variables and use the parabolic symmetrizer derived in [14] to get the linear, constant coefficient and symmetric system

$$w_t + Aw_x + Bw_y = \varepsilon\Big((C_{11}w_x + C_{12}w_y)_x + (C_{21}w_x + C_{22}w_y)_y\Big), \tag{6}$$

where the symmetrized variables are

$$w = \left[\frac{\bar{c}}{\sqrt{\gamma}\bar{\rho}}\rho, u, v, \frac{1}{\bar{c}\sqrt{\gamma(\gamma-1)}}T\right]^T. \tag{7}$$

All matrix coefficients can be found in [14] but we restate them in Appendix A for convenience.

We will use the energy method to determine the well-posedness of (6). The energy norm in which we will derive our estimates is defined by

$$\|w\|^2 = \int_\Omega w^T w \, d\Omega. \tag{8}$$