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# A rescaling velocity method for dissipative kinetic equations. Applications to granular media

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ABSTRACT

We present a new numerical algorithm based on a relative energy scaling for collisional kinetic equations allowing to study numerically their long time behavior, without the usual problems related to the change of scales in velocity variables. It is based on the knowledge of the hydrodynamic limit of the model considered, but is able to compute solutions for either dilute or dense regimes. Several applications are presented for Boltzmann-like equations. This method is particularly efficient for numerical simulations of the granular gases equation with dissipative energy: it allows to study accurately the long time behavior of this equation and is very well suited for the study of clustering phenomena.



## 1. Introduction

We are interested in numerical simulations of the long time behavior of collisional kinetic equations such as the Boltzmann equation for granular gases. To this end, we introduce a new technique which is based on the information provided by the hydrodynamic fields computed from a macroscopic model corresponding to the original kinetic equation. Then we simply rescale the kinetic equation according to these hydrodynamic quantities. The reason to do so is that the change of scales in velocity is a challenging numerical problem when one wants to deal with solutions to dissipative kinetic equations on a fixed grid. Indeed, most of the usual deterministic methods fails to capture the correct long time behavior because of concentration or spreading over the velocity space.

Recently, Filbet and Russo proposed in [25] a rescaling method for space homogeneous kinetic equations on a fixed grid. This idea is mainly based on the self-similar behavior of the solution to the kinetic equation. However for space non homogeneous case, the situation is much more complicated and this method cannot be applied since the transport operator and the boundary conditions break down this self-similar behavior. Here we propose a very simple idea based on a rescaling of the kinetic equation according to its hydrodynamic limit. We will see that this approach can actually be applied to various type of collisional kinetic equations such as elastic and inelastic Boltzmann equation (also known as the *granular gases equation*).

Among popular methods for numerical simulations of collisional gases, the most widely used are meshless. On the one hand, the Molecular Dynamics algorithm is a deterministic method up to the random initialization of the particles which is valid in a dense regime [32]. This method has been used to describe the apparition of shocks in supersonic sand and reproduces very accurately experimentation results [41]. On the other hand, Direct Simulation Monte Carlo (DSMC) methods are stochastic algorithms working either in dense or rarefied regimes. We refer for instance to [5] for a complete review of this

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topic. These methods are really efficient in term of computational cost since their complexity grows linearly with the number $N$ of particles but they are rather inaccurate since the order of accuracy is about $\mathcal{O}(1/\sqrt{N})$. Another approach consists in a direct resolution of the Boltzmann operator on a phase space grid. For instance, deterministic and highly accurate methods based on a spectral discretization of the collisional operator have been proposed by F. Filbet, G. Naldi, L. Pareschi, G. Toscani and G. Russo in [39,37,24] for the space-homogeneous setting. Although being of complexity $\mathcal{O}(N^2)$, they are spectrally accurate and then need very few points to be precise. Another spectral method, inspired of the direct fast Fourier transform approach of A.V. Bobylev and S. Rjasanow [8,10,9] using Lagrange multiplier to improve the conservations was also used for space-homogeneous simulations in 3D by Gamba and Tharkabhushanam in [28].

Due to the large number of particles involved in the study of rarefied gas dynamics, we shall adopt a statistical physics' point of view, by the use of Boltzmann-like kinetic equations. For a given nonnegative initial condition $f_0$, we will consider a particle distribution function $f^\varepsilon = f^\varepsilon(t,x,v)$, for $t \geqslant 0, x \in \Omega \subset \mathbb{R}^{d_x}$ and $v \in \mathbb{R}^{d_v}$, solution to the initial-boundary value problem

$$\begin{cases} \frac{\partial f^\varepsilon}{\partial t} + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon}\, \mathcal{Q}(f^\varepsilon, f^\varepsilon), \\ f^\varepsilon(0,x,v) = f_0(x,v). \end{cases} \tag{1.1}$$

where the collision operator $\mathcal{Q}$ is a Boltzmann-like operator, which preserves at least mass and momentum. This equation describes numerous models such as the Boltzmann equation for elastic and inelastic collisions or Fokker–Planck–Landau type equations. The parameter $\varepsilon > 0$ is the dimensionless Knudsen number, that is the ratio between the mean free path of particles before a collision and the length scale of observation. It measures the rarefaction of the gas: the gas is *rarefied* if $\varepsilon \sim 1$ and *dense* if $\varepsilon \ll 1$. The open set $\Omega$ is a bounded Lipschitz-continuous domain of $\mathbb{R}^{d_x}$, which means that the model (1.1) has to be supplemented with boundary conditions described later.

The article is organized as follows. We present in Section 2 the rescaling velocity method and give two possible choices of scaling functions, the coupled Filbet-Russo scaling and the new uncoupled macroscopic one. We choose to use the latter in the rest of the article, and apply it to three different collision operators in Section 3: the Boltzmann operator, the full granular gases operator and a simplified BGK-like granular gases operator. Subsequently, we introduce in Section 4 the spatial boundary conditions we have to impose on $\Omega$: the so-called *Maxwell* boundary conditions, a convex combination of specular and diffusive reflections at the boundary. We present these conditions for the kinetic model (in classical and rescaled variables) and for the macroscopic equations. In Section 5, we introduce the numerical methods we shall use for the discretization of each term of the problem: the collision operator, the conservative transport term and the system of conservation laws with source term. Finally, we present in Section 6 some numerical results concerning the full and simplified granular gases models, in space homogeneous and inhomogeneous settings, with different geometries.

## 2. The rescaling velocity method

This section is devoted to the presentation of a new scaling for Eq. (1.1) allowing to follow the change of scales in velocity. It is an extension to the space-dependent setting of the method first introduced in [25], using the relative kinetic energy as a scaling function. This method was reminiscent from the work of Bobylev, Carrillo, and Gamba [7] about Enskog-like inelastic interactions models. Indeed, in one section of this work, the authors scaled the solution of the space homogeneous collision equation by its thermal velocity, in order to study a drift-collision equation, where no blow-up occurs. The same technique was also used by Mischler and Mouhot in [35] to prove the existence of self-similar solutions to the granular gases equation. This temperature-dependent scaling has also been introduced by Carrillo and Toscani in [15] and Carrillo, Di Francesco and Toscani in [13] for nonlinear diffusion equations. It was finally extended by Carrillo and Vázquez in [16] to show the apparition of chaotic behavior for a precise (constructive) nonlinearity: the self-similar profile of this equation "oscillates" between Gaussian (heat equation) and Zel'dovich–Kompaneets–Barenblatt (porous medium equation) profiles.

Let us drop for simplicity the $\varepsilon$-dependence in $f$. For a given positive function $\omega : \mathbb{R}^+ \times \Omega \mapsto \mathbb{R}^+$, we introduce a new distribution $g(t,x,\xi)$ by setting

$$f(t,x,v) = \frac{1}{\omega^{d_v}} g\Big(t,x,\frac{v}{\omega}\Big), \tag{2.1}$$

where the function $\omega$ is assumed to be an accurate measure of the "support" or scale of the distribution $f$ in velocity variables. Then according to this scaling, the distribution $g$ should naturally "follow" either the concentration or the spreading in velocity of the distribution $f$.

Let us now derive the kinetic equation verified by the distribution $g$. Differentiating relation (2.1) with respect to time yields

$$\frac{\partial f}{\partial t} = \frac{1}{\omega^{d_v}} \left[ \frac{\partial g}{\partial t} - \frac{1}{\omega} \frac{\partial \omega}{\partial t} \operatorname{div}_\xi(\xi g) \right].$$

Provided that $v = \omega \xi$, one also has