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The combined Lagrangian advection method

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ABSTRACT

We present and test a new hybrid numerical method for simulating layerwise-two-dimensional geophysical flows. The method radically extends the original Contour-Advective Semi-Lagrangian (CASL) algorithm [5] by combining *three* computational elements for the advection of general tracers (e.g. potential vorticity, water vapor, etc.): (1) a pseudo-spectral method for large scales, (2) Lagrangian contours for intermediate to small scales, and (3) Lagrangian particles for the representation of general forcing and dissipation. The pseudo-spectral method is both efficient and highly accurate at large scales, while contour advection is efficient and accurate at small scales, allowing one to simulate extremely fine-scale structure well below the basic grid scale used to represent the velocity field. The particles allow one to efficiently incorporate general forcing and dissipation.

1. Introduction

Numerical simulations of atmospheric and oceanic fluid dynamics must deal with a vast range of active spatial and temporal scales of motion. Much of this range lies beyond the limit of numerical resolution, requiring small scales (and high frequencies) to be instead parametrised by 'eddy-diffusivity' or 'closure' schemes meant to approximate the collective effects of unresolved motions on resolved ones (cf. [17] and references). Here, we describe a new modelling advance which reduces the need for closure schemes by allowing one to efficiently extend the range of resolved scales, in particular for advected tracers. The new advance is the culmination of years of model development based on "Contour Advection" (CASL, [5]), a hybrid Lagrangian–Eulerian method stemming originally from "Contour Surgery" [4] and "Contour Dynamics" [20]. The new method, called the "Combined Lagrangian Advection Method" (CLAM), utilises three computational elements—contours, particles, and grid points (or spectral coefficients)—combined in a way to optimise performance and accuracy.

While CLAM is built for accurate conservation in the absence of forcing and dissipation, it also allows one to efficiently handle general non-conservative processes such as thermal heating, Ekman friction, and stochastic forcing [7,16,13]. Moreover, it may offer distinct advantages over commonly-used numerical methods in Geophysical Fluid Dynamics when multiple tracers (dynamical, chemical, biological) are considered.

In Section 2 below, we outline the structure of the method. It is next illustrated and tested in an example of forced two-dimensional turbulence in Section 3. Then it is applied to study an aspect of the banded circulation patterns found in planetary atmospheres in Section 4. Conclusions and ideas for further model development are offered in Section 5.

2. The method

CLAM was developed originally to better model both unforced and forced 2D turbulence at ultra-high Reynolds numbers [9–11,13]. It is an extension of the recent HyperCASL algorithm [13], which introduced the idea of using point vortices or

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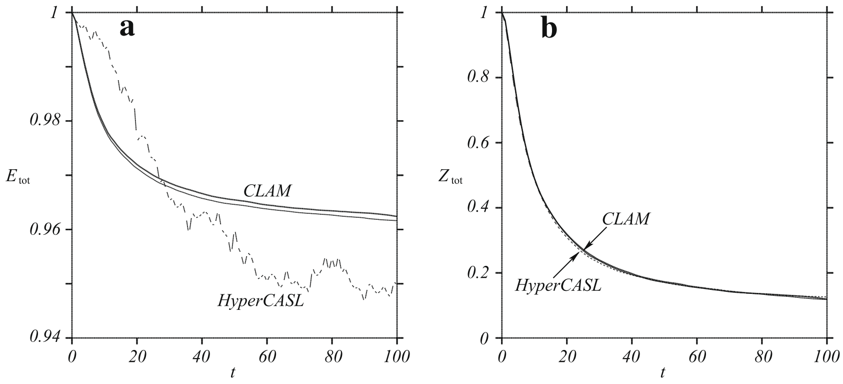

Fig. 1. Comparison of the evolution of (a) energy $E_{\text{tot}}(t)$ (normalised by $E_{\text{tot}}(0) = 0.0148588$) and (b) enstrophy $Z_{\text{tot}}(t)$ (normalised by $Z_{\text{tot}}(0) = 4.046645$) between HyperCASL (dashed) and CLAM (solid) in the case of freely-decaying two-dimensional turbulence examined in Fontane and Dritschel [13]. The two curves shown for CLAM correspond to two different ways of representing q_d, by particles (bold) or by a spectral method (thin). See text for details.

particles to represent a *residual* tracer field q_d (e.g. vorticity in 2D turbulence or potential vorticity in rotating stratified flows). The residual tracer field q_d is used as a temporary reservoir for any explicit forcing and dissipation $S(\boldsymbol{x}, t)$ operating on the full tracer field q:

$$\frac{\mathrm{D}q}{\mathrm{D}t} = S(\boldsymbol{x}, t) \tag{1}$$

At any instant of time t, the full tracer field q is the sum

$$q = q_a + q_d \tag{2}$$

in which q_a evolves *conservatively* by contour advection, i.e.

$$\frac{\mathrm{d}\boldsymbol{X}_a}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{X}_a, t) \tag{3}$$

(equivalent to $\mathrm{D}q_a/\mathrm{D}t = \partial q_a/\partial t + \boldsymbol{u}\cdot\nabla q_a = 0$), where $\boldsymbol{X}_a$ is a point on a contour and $\boldsymbol{u}(\boldsymbol{x}, t)$ is the velocity field, while q_d evolves by advecting discrete particles $\boldsymbol{X}_d$,

$$\frac{\mathrm{d}\boldsymbol{X}_d}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{X}_d, t) \tag{4}$$

as well as adjusting the intensities Γ_d of individual particles to match the imposed forcing and dissipation (see Appendix A). Every 4 eddy-turnaround times T_{eddy} (determined from the maximum vorticity integrated over time), q_d is transferred to a set of contours representing the *primary* tracer field q_a through an efficient contouring procedure [7]. Standard bi-linear interpolation is used to create a corresponding gridded field of q_d from the particles when needed (see Appendix A).

Weak numerical dissipation occurs during the regularization of contours by "surgery" [4], and when resetting the particles on a regular array. Both are done approximately every $0.2T_{\text{eddy}}$. Notably, in simulations of freely-decaying two-dimensional turbulence, it has been shown that this dissipation is comparable to that needed in a conventional spectral method using a grid 10–20 times finer than used in HyperCASL and CLAM to represent the velocity field $\boldsymbol{u}$ [11].

An unwanted feature of HyperCASL is the introduction of a small level of numerical error or stochastic noise in q_a by the contour-to-grid conversion procedure and, to a much lesser extent, by contour surgery [13]. Unfortunately, this noise is statistically uniform across Fourier modes and it generates a growing k^1 tracer variance spectrum at small k. In simulations of 2D turbulence [13], this gives rise to a growing k^{-1} energy spectrum at small k (the actual energy spectrum normally decays rapidly as $k \to 0$, see e.g. [10]).[1] As a consequence, this noise primarily affects the energy while enstrophy (vorticity variance) is more robust, see Figs. 1 and 2. And in long-time simulations, it can eventually lead to significant erroneous energy loss.

CLAM removes this unwanted feature (see Fig. 1(a)) by using a pseudo-spectral (PS) method to model large scales, specifically wavenumbers $k \leqslant k_c$, where k_c is the 'filter cutoff wavenumber'. The PS method is well-designed for this purpose and, moreover, is both accurate and efficient. In CLAM, the tracer field computed this way, denoted q_s, is blended with the primary tracer field q_a (represented by contours). The full tracer field is obtained now from

$$\hat{q} = F\hat{q}_s + (1 - F)\hat{q}_a + \hat{q}_d \tag{5}$$

where a hat denotes a spectral transform, and $F(k)$ is a low-pass filter (see below). The only difference between HyperCASL and CLAM is the replacement of q_a in (2) by a weighted sum of q_s and q_a. The form of the filter $F(k)$ was fixed after extensive numerical tests (see below), and it takes the form

[1] See Fig. 8 in the online notes at www-vortex.mcs.st-and.ac.uk/HyperCASL.pdf accompanying Fontane and Dritschel [13].