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An efficient compact difference scheme for solving the streamfunction formulation of the incompressible Navier–Stokes equations

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ABSTRACT

Recently, a new paradigm for solving the steady Navier–Stokes equations using a streamfunction–velocity formulation was proposed by Gupta and Kalita [M.M. Gupta, J.C. Kalita, A new paradigm for solving Navier–Stokes equations: streamfunction–velocity formulation, J. Comput. Phys. 207 (2005) 52–68], which avoids difficulties inherent in the conventional streamfunction–vorticity and primitive variable formulations. It is discovered that this formulation can reached second-order accurate and obtained accuracy solutions with little additional cost for a couple of fluid flow problems.

In this paper, an efficient compact finite difference approximation, named as *five point constant coefficient* second-order compact (5PCC-SOC) scheme, is proposed for the streamfunction formulation of the steady incompressible Navier–Stokes equations, in which the grid values of the streamfunction and the values of its first derivatives (velocities) are carried as the unknown variables. The derivation approach is simple and can be easily used to derive compact difference schemes for other similar high order elliptic differential equations. Numerical examples, including the lid driven cavity flow problem and a problem of flow in a rectangular cavity with the hight–width ratio of 2, are solved numerically to demonstrate the accuracy and efficiency of the newly proposed scheme. The results obtained are compared with ones by different available numerical methods in the literature. The present scheme not only shows second-order accurate, but also proves more effective than the existing second-order compact scheme of the streamfunction formulation in the aspect of computational cost.

1. Introduction

Numerical solution of incompressible Navier–Stokes equations is the most significant area in computational fluid dynamics (CFD) related fields in science and engineering. The two-dimensional (2D) incompressible flows are usually solved with the conventional primitive variable or the streamfunction–vorticity formulation of the Navier–Stokes equations. Over the past couple of decades, these formulations have been utilized by a large number of authors to test methods proposed for solving numerically a variety of fluid flow or heat transfer problems [2,3,5–20,22]. The primitive variable form of Navier–Stokes equations can accurately describe the fluid flow phenomena, but a major difficult comes from the pressure term in the momentum equations and must be implicitly updated for the incompressibility to be satisfied. The streamfunction–vorticity form of Navier–Stokes equations has been used to avoid handling the pressure

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variable for several decades. The vorticity formulation is obtained by taking the rotational of the momentum equations. The difficulty with the vorticity formulation is the lack of the simple physical boundary conditions for the vorticity field at the no-slip boundaries. When the vorticity transport equation is numerically solved, as mentioned by some researchers [27,28,45], a variety of numerical approximations for the boundary values of vorticity has to be carried out. In [45], a method of vorticity–streamfunction dynamics has been developed by Ben-Artzi, Fishelov and Trachtenberg. An important corollary of this method is that the difficulty of determining the vorticity on the boundary was already avoided. Later, in order to circumvent the pitfalls associated with the vorticity values at the boundary, the streamfunction–velocity or the pure streamfunction formulation-based methodology for the solution of the 2D incompressible fluid flows has been utilized by some authors [27–31,46,47]. The main advantage of this type of formulation is that the boundary conditions of streamfunction and velocity are generally known and are easy to implement computationally, thus the computational schemes proposed are very efficient and accurate. The scheme developed in this paper belongs to this category.

We consider the following streamfunction–velocity formulation of the 2D steady Navier–Stokes equations representing incompressible fluid flows:

$$\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}=Re\left[u\left(\frac{\partial^3\psi}{\partial x^3}+\frac{\partial^3\psi}{\partial x\partial y^2}\right)+v\left(\frac{\partial^3\psi}{\partial x^2\partial y}+\frac{\partial^3\psi}{\partial y^3}\right)\right] \tag{1}$$

$$u=\psi_y,\quad v=-\psi_x \tag{2}$$

where ψ is the streamfunction, u and v are the velocities and Re is the nondimensional Reynolds number.

When the streamfunction–velocity formulation (1), which is a fourth-order partial differential equation, is solved by finite differences, a uniform grid with 13 points must be needed to obtain a classical second-order finite difference (FD) approximation. This difference discretization using 13 grid points needs to be amended at grid points near the boundaries and must bring about difficulties for the solution of the resulting linear systems [27]. In order to overcome the above drawback, Gupta and Kalita [27] proposed a new paradigm for solving the steady Navier–Stokes equations: streamfunction–velocity formulation and derived two second-order compact finite difference schemes that carry streamfunction and its first derivatives (velocities) as the unknown variables for this equation using the *Mathematica* code. They used a biconjugate gradient method to obtain the numerical solutions of the fluid flow problems and solved the cavity flow with a grid size of 161×161 for $3200\leqslant Re\leqslant 10{,}000$ and a problem of flow in a rectangular cavity with the hight–width ratio of 2 with a grid size of 81×161 for $100\leqslant Re\leqslant 1500$. Similar a second-order compact finite difference formulation for the streamfunction–velocity formulation (1) was presented by Tian et al. [33]. They used a multigrid method to obtain the numerical solutions for the lid driven cavity flow problem for Reynolds numbers up to $Re = 3200$ with a maximum of 257×257 grid mesh.

In Fig. 1, we denote the placement of nine points, and number the mesh points (x_i,y_j), (x_i+h,y_j), (x_i,y_j+h), (x_i-h,y_j), (x_i,y_j-h), (x_i+h,y_j+h), (x_i-h,y_j+h), (x_i-h,y_j-h) and (x_i+h,y_j-h) as 0, 1, 2, 3, 4, 5, 6, 7 and 8, respectively, where h denotes the mesh size of x- and y-directions. In writing the finite difference approximations a single subscript 'k' denotes the corresponding function value at the mesh point numbered 'k'.

Using the *Mathematica* code, Gupta and Kalita [27] proposed two second-order compact difference approximations for the solution of the streamfunction formulation (1), which are given by

$$28\psi_0-8\sum_{k=1}^{4}\psi_k+\sum_{k=5}^{8}\psi_k=3h(v_1-v_3-u_2+u_4)+\frac{1}{2}Reh^2\left(v_0\sum_{k=1}^{4}u_k-u_0\sum_{k=1}^{4}v_k\right) \tag{3}$$

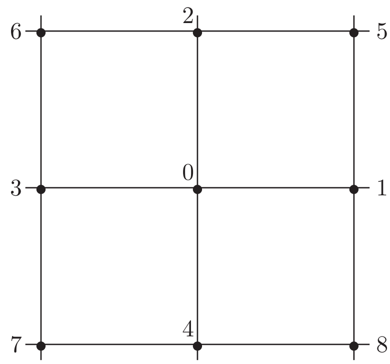

Fig. 1. Computational stencil.