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A multislope MUSCL method on unstructured meshes applied to compressible Euler equations for axisymmetric swirling flows

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ABSTRACT

A finite volume method for the numerical solution of axisymmetric inviscid swirling flows is presented. The governing equations of the flow are the axisymmetric compressible Euler equations including swirl (or tangential) velocity. A first-order scheme is introduced where the convective fluxes at cell interfaces are evaluated by the Rusanov or the HLLC numerical flux while the geometric source terms are discretizated to provide a well-balanced scheme i.e. the steady-state solutions with null velocity are preserved. Extension to the second-order space approximation using a multislope MUSCL method is then derived. To test the numerical scheme, a stationary solution of the fluid flow following the radial direction has been established with a zero and nonzero tangential velocity. Numerical and exact solutions are compared for classical Riemann problems where we employ different limiters and effectiveness of the multislope MUSCL scheme is demonstrated for strongly shocked axially symmetric flows like in spherical bubble compression problem. Two other tests with axisymmetric geometries are performed: the supersonic flow in a tube with a cone and the axisymmetric blunt body with a free stream.

1. Introduction

Axisymmetric Euler system using cylindrical coordinates is used in numerous applications such as axisymmetric flows in a nozzle [14], supersonic jets [15], turbo machine modeling [11,23]. More recently, inductive plasma flows are modeled with the axisymmetric Euler formulation taking into account the tangential velocity, the so-called swirling flow [20,24]. An axisymmetric formulation avoids considering a full three-dimensional problem leading to a strong computational cost reduction and meshes are easier to generate.

From a numerical point of view, the finite volume method [6,8,10] is a popular technique to compute numerical approximations of the Euler system solution for axisymmetric geometries but most of the authors have neglected the swirl velocity which is of crucial importance in some applications such as the inductive plasma problem. A particular issue concerns the choice of the variables to conserve. In a first approach, the mean value approximation of any generic function v on cell C_i is performed by using the classical average

$$v_i \approx \frac{\int_{C_i} v\,drdz}{\int_{C_i} drdz}$$

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where we employ the measure (metric) $dr\,dz$ [5]. A second approach consists in computing the mean value using the measure $r\,dr\,dz$ [10]:

$$v_i \approx \frac{\int_{C_i} vr\,drdz}{\int_{C_i} r\,drdz}$$

which leads to a better formulation. Indeed, v_i corresponds to the mean value of v in the three-dimensional context, i.e., the mean value on the axisymmetric torus with section C_i. On the other hand, the formulation simplifies the boundary condition at $r = 0$ since a null flux value naturally derives from the flux integration and no additional constraint is required on the symmetry axis [10].

To provide an approximation of the solution of the axisymmetric Euler system, we use a fractional step technique where we split the formulation into a conservative homogeneous equation and the source term. To solve the conservative part, a standard technique consists in using a first-order solver (Rusanov, HLLC, Roe or Riemann solver) combined with a second-order reconstruction such as the MUSCL method [13,21] to improve accuracy. The classical MUSCL technique uses a piecewise linear reconstruction with a slope limiting procedure to ensure L^∞-stability. Then two new approximations are computed on both sides of each edge and are employed in the numerical flux evaluation. We propose here to use a new reconstruction technique: the multislope MUSCL method [2–4] where the reconstructed values are obtained using an approximation of specific directional derivatives instead of the full gradient. The main advantage is that the reconstruction can be rewritten as a one-dimensional MUSCL method at each interface leading to a simple and efficient scheme.

In the axisymmetric context, there are few numerical tests to validate the scheme for compressible Euler equations. For example, we are not able to compute the exact solution of the Riemann problem excepting in very particular situations. We propose a new numerical test for the swirling flow based on the steady-state situation. We manage to reduce the Euler system to an ordinary differential equation and a simple approximation based on the forward Euler scheme is proposed to provide an accurate numerical solution.

The organization of the paper is as follows. In Section 2, we present the governing Euler equations in cylindrical coordinates assuming rotational symmetry. In Section 3, we present the numerical scheme and its second-order extension using a multislope MUSCL method. In Section 4, we establish a stationary solution assuming that the flow depends only on the radial direction and that the axial velocity is null. Finally, we present numerical experiments to test the obtained scheme.

2. Axisymmetric Euler equations for swirling flows

We first present the compressible Euler equations using the cylindrical coordinates and simplify them under the axisymmetric invariance assumption. For any point $X = (x, y, z) \in \mathbb{R}^3$ we denote by (r, θ, z) the associated cylindrical coordinates. Let

$$\mathcal{P} = \mathbb{R}^+ \times \mathbb{R} = \{(r,z) \in \mathbb{R}^2;\ r \geqslant 0\}$$

denote an axial cut of the three-dimensional space (the set of parameters) and let Ω be an open set of $\mathcal{P}$. The open set $\widetilde{\Omega} \subset \mathbb{R}^3$ will denote the three-dimensional volume obtained by rotation around the axial direction $0z$, i.e.

$$\widetilde{\Omega} := \{(r\cos\theta, r\sin\theta, z);\ (r,z) \in \Omega,\ 0 \leqslant \theta < 2\pi\}.$$

We start by giving the compressible Euler equations in the domain $\widetilde{\Omega}$:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + P\mathbf{I}) = 0, \tag{2}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot ((E + P)\mathbf{u}) = 0, \tag{3}$$

where ρ is the fluid mass density, P is the pressure, $\mathbf{u}$ is the velocity vector and E is the total energy per unit volume. The tensors $\mathbf{u} \otimes \mathbf{u}$ and $\mathbf{I}$ stand for the tensor product of $\mathbf{u}$ by $\mathbf{u}$ and the unit tensor respectively.

To close the system, we add a state equation which in general form reads

$$P = \hat{P}(\rho, e),$$

where e stands for the specific internal energy related to the total energy by

$$E = \rho e + \frac{1}{2}\rho|\mathbf{u}|^2.$$

In the following, we restrict ourselves to an ideal gas, that is,

$$P = (\gamma - 1)\rho e, \tag{4}$$

where γ is the ratio of specific heats at constant pressure and volume.