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Variational piecewise constant level set methods for shape optimization of a two-density drum

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ABSTRACT

We apply the piecewise constant level set method to a class of eigenvalue related two-phase shape optimization problems. Based on the augmented Lagrangian method and the Lagrange multiplier approach, we propose three effective variational methods for the constrained optimization problem. The corresponding gradient-type algorithms are detailed. The first Uzawa-type algorithm having applied to shape optimization in the literature is proven to be effective for our model, but it lacks stability and accuracy in satisfying the geometry constraint during the iteration. The two other novel algorithms we propose can overcome this limitation and satisfy the geometry constraint very accurately at each iteration. Moreover, they are both highly initial independent and more robust than the first algorithm. Without penalty parameters, the last projection Lagrangian algorithm has less severe restriction on the time step than the first two algorithms. Numerical results for various instances are presented and compared with those obtained by level set methods. The comparisons show effectiveness, efficiency and robustness of our methods. We expect our promising algorithms to be applied to other shape optimization and multiphase problems.

1. Introduction

Optimal shape design is very important and challenging in science and engineering. The typical problem is to find the optimal shape that minimizes or maximizes an objective functional satisfying certain PDE and geometry constraints. The essential difficulty for solving shape optimization problems is that the topology of the optimal shape is unknown *a priori*. One therefore needs to find a mechanism to represent a shape and follow its evolution. Moreover, the changing topology should be handled automatically during the evolution.

One type of such problems arises from structural vibration control and has many engineering design applications [5,42], such as the band structure optimization of photonic crystals [12,16,17]. As a model problem, we consider an acoustic drum head with a fixed bounded domain $\Omega \subset \mathbb{R}^2$ and variable density $\rho(x)$. The resonant frequencies of the drum satisfy the following eigenvalue problem:

$$\begin{cases} -\Delta u(x) = \lambda\rho(x)u(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Let $S \subset \Omega$ be an unknown domain. Suppose that the density $\rho(x)$ is a piecewise constant function satisfying

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$$\rho(x) = \begin{cases} \rho_1, & \text{if } x \in \Omega \setminus S, \\ \rho_2, & \text{if } x \in S. \end{cases} \tag{2}$$

The mathematical problem we investigate here is to find an optimal distributed parameter $\rho(x)$ that solves the following constrained optimization problem:

$$\min \lambda_1 \text{ or } \max \lambda_1 \text{ or } \max (\lambda_{m+1} - \lambda_m), \quad m = 1, 2, 3, 4, \tag{3}$$

subject to

$$\|S\| = K, \tag{4}$$

where $\|\cdot\|$ denotes the area of a domain and K is some prescribed number.

Osher and Santosa [37] firstly solved the above model problem elegantly by the level set method (LSM). They proposed an effective algorithm by combining the variational LSM [59] and the projection gradient method [43]. They used the Lagrange multiplier technique to convert the constrained optimization problem to an unconstrained one. At each iteration, the Lagrange multiplier was solved using a projection approach based on the linearization of the constraint. When an iteration has violated the constraint by a prescribed tolerance, they found an optimal Lagrange multiplier using Newton's method.

The LSM originally proposed by Osher and Sethian [38] is a versatile tool when dealing with problems required tracing interfaces separating a domain into subregions. The interface is represented implicitly by the zero level set of a Lipschitz continuous level set function (LSF). In the conventional LSM, the Hamilton–Jacobi equation for the LSF is solved to evolve the interface by using a capturing Eulerian approach. Upwind schemes, higher order essentially non-oscillatory (ENO) [39] and weighted ENO (WENO) [25] schemes can be used to solve this equation. During the evolution, often regularity is imposed on the LSF by requiring it to be a signed distance function. The so-called re-initialization process should be performed periodically. There are some effective numerical approaches for re-initialization, such as the fast marching method [44] and the fast sweeping algorithm [55]. LSMs can easily handle certain types of shape and topological changes, such as merging, splitting and developing sharp corners. The methods can therefore be naturally used to solve optimal shape design and topology optimization problems [3,46,57]. For more details about the LSMs and their wide applications, we refer to see [9,11,13,19,35,36,45,50,51] and the references therein.

Motivated by the level set ideas in [37,27], Haber [22] minimized and maximized the first eigenvalue by using a reduced Hessian sequential quadratic programming method combined with multilevel continuation techniques. Strang and Persson [41] solved the eigenvalue problem (1) in an irregular domain using the finite element method on unstructured meshes generated by the LSM [40]. Then they used the gradient descent method of Osher and Santosa [37] to minimize the first and the second eigenvalue. Their method can work with arbitrary domains and resolve the interface by the triangular mesh, but remeshing is needed at each iteration. Recently, Brandman [7] used the LSM to compute the eigenvalues of an elliptic operator defined on a hypersurface, which is represented implicitly as the zero level set of a LSF.

In classical shape sensitivity analysis for shape optimization problems, shape derivatives that measure the sensitivity of boundary or interface perturbations are derived to obtain the shape gradient of the objective functional. For detailed theoretical analysis and applications of shape sensitivity analysis, we refer to see [48] and the references therein. After calculation of the shape gradient, gradient-type algorithms are used to decrease the objective functional and stop if the shape gradient vanishes. But such methods tend to fall into local minima and are generally implemented under the Lagrangian framework which requires remeshing at each iteration. The homogenization method [1] can overcome the two drawbacks, but it is mainly restricted to linear elasticity and gives optimal shapes that are composite. Penalization methods are needed to project the composite shape on a classical two-phase design.

The combination of LSMs with the shape sensitivity analysis framework has become a standard tool for solving a variety of shape optimization and inverse problems in engineering [2,3,8,9,15,57,60]. Fast Newton-type shape optimization methods were used for level set formulations in [24]. However, as pointed out in [2,3,10,20,23,56], the conventional LSM based on shape sensitivity analysis cannot create new holes automatically during the evolution which may get stuck at shapes with fewer holes than the optimal geometry. Therefore, the initial shape guess generally contains many holes. To eliminate this weakness, topology derivatives were incorporated into shape derivatives based LSMs for inverse obstacle problems [10], structure optimization [4] and shape optimization problems [20]. The topology derivative introduced firstly by Sokołowski and Żochowski [47] measures the influence of creating small holes centered at a certain point in the domain. Motivated by the idea in [10], He et al. [23] combined shape derivatives with topological derivatives in LSMs to maximize band gaps and presented an algorithm more efficient and flexible in topology changing than the original LSM based on the shape derivatives. For optimizing spectral gaps of the drum, they improved the numerical results in [37].

As recent variants of the standard LSMs, piecewise constant level set methods (PCLSMs) were proposed by Tai et al. [31,52] for image segmentation and elliptic inverse problems [53]. Similar ideas can be found in [21,30,33,49]. The systematic and general framework of the PCLSM was presented in [29]. Different from standard LSMs, the PCLSM can identify subregions using one discontinuous piecewise constant level set function (PCLSF) which can only take piecewise constant values at convergence. One has to employ N LSFs to represent up to 2^N subregions in the LSM, while PCLSM can use one LSF to distinguish multiple regions. The LSM propagates the interface by defining speed only on the interface, which makes it generally could not create small holes at the places far away from the interface. The PCLSM determines the interface by forcing the value of the LSF at each mesh point to be one of the piecewise constant values. Therefore, solving shape and topology