Journal of Computational Physics 247 (2013) 1–16

Contents lists available at SciVerse ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Mimetic discretization of two-dimensional magnetic diffusion equations

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ARTICLE INFO

Article history:
Received 22 June 2012
Received in revised form 30 January 2013
Accepted 25 March 2013
Available online 15 April 2013

Keywords:
Mimetic finite differences
Magnetic diffusion
Cylindrical coordinates
Unstructured mesh

ABSTRACT

In case of non-constant resistivity, cylindrical coordinates, and highly distorted polygonal meshes, a consistent discretization of the magnetic diffusion equations requires new discretization tools based on a discrete vector and tensor calculus. We developed a new discretization method using the mimetic finite difference framework. It is second-order accurate on arbitrary polygonal meshes and a consistent calculation of the Joule heating is intrinsic within it. The second-order convergence rates in L^2 and L^1 norms were verified with numerical experiments.

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1. Introduction

Reliability and robustness of hydrodynamic codes, especially Lagrangian codes, is based on availability of accurate numerical schemes that work on highly distorted unstructured meshes. These codes require magneto-hydrodynamic (MHD) capabilities in order to model a larger set of validation experiments. In this paper, we develop a second-order discretization method for the magnetic diffusion equations in cylindrical coordinates.

In the case of an azimuthal magnetic field, $\mathbf{B}(r,\phi,z) = B(r,z)\,\boldsymbol{\phi}$ the equations of magnetic diffusion are reduced to the parabolic equation for rB:

$$\frac{\partial B}{\partial t} = \frac{\partial}{\partial r}\left(\frac{\mathbb{K}}{4\pi r}\frac{\partial (rB)}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{\mathbb{K}}{4\pi r}\frac{\partial (rB)}{\partial z}\right),$$

where $\mathbb{K}$ is a scalar. Written in this form, the equation is amenable to discretizations using existing diffusion schemes; however, attempts to extend these schemes ran into various numerical issues on unstructured polygonal meshes that are typical for Lagrangian hydrodynamics. For instance, we do not know any scheme that is either exact for the linear solution $B = r$ or accurate enough near the axes $r = 0$. The underlying reason is that the right-hand side represents the elliptic operator with respect to the Cartesian coordinate r and z; thus, special attention should be taken to a proper approximation of the singular coefficient $\mathbb{K}/r$.

The discretization task becomes even more complex in the case of highly distorted meshes, especially polygonal meshes. Derivation of a consistent or accurate scheme needs to use tools of a discrete vector and tensor calculus to preserve important properties of the continuum equation. This includes exactness for linear solutions B and symmetry and positivity of a discrete operator. In this paper, we employ the mimetic finite difference framework.

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0021-9991/$ - see front matter Published by Elsevier Inc.
http://dx.doi.org/10.1016/j.jcp.2013.03.050

The mimetic finite difference (MFD) method [7,6] is a significant generalization of mimetic methods developed in 1990s (see [11,9] and references therein) and the support-operator (SO) methods developed in 1980s by the group of Samarskii et al. [20]. In the SO method, the first principle (the Stokes theorem) is used to define a discrete approximation of a first-order differential operator, such as the divergence, gradient, or curl. This initial discrete operator, called the primary operator, supports the construction of the other discrete operator, called the derived operator, using a discrete integration by parts formula. The discrete calculus based on primary and derived operators has analogs of the exact identities (e.g. $\mathrm{div}\,\mathrm{curl} = 0$) and the Helmholtz decomposition theorems. For 3D Maxwell's equations, these and other properties lead to energy conservation and preservation of a divergence-free condition in the mimetic schemes [23]. The modern MFD method changes the way how the derived operators are constructed which lead to more accurate schemes that remain consistent on arbitrary polygonal and polyhedral meshes. Compared to the previous work, new approach provides a simple description of a rich family of schemes, a convergence theory, and applicability to a larger set of meshes.

The mimetic methods mimic or preserve essential mathematical and physical principles in discrete models. These principles include (but are not limited to) discrete vector and tensor calculus [10], discrete maximum principles [13], and discrete conservation laws [21]. Importance of having a rich family of the discretization schemes with equivalent properties becomes evident during building a monotone mimetic scheme. In some cases, mimetic discretization methods can be related to other compatible discretization methods such as finite elements [19], finite volumes [8], and summation by parts [17] methods.

The practical implementation of the MFD method is relatively simple on arbitrary polygonal and polyhedral meshes. Consider, for example, a diffusion problem. In a finite volume method, the fluxes are defined only at interfaces between mesh elements and a finite difference formula is used to discretize the constitutive equation, e.g. Darcy's or Fick's law. In contrast, in the mixed finite element method, a polynomial representation of the vector field inside each mesh element is used to define the inner product between vectors and then to write the constitutive equation by duality. This, however, can be done only for simple geometrical shapes. The MFD method combines flexibility of finite volume meshes with power of the finite element analysis. In the MFD method there is a notion of the inner product between vectors but the vector field inside a mesh element is never introduced. This flexibility is related directly to the existence of a family of schemes.

In the last decade, a few MFD methods have been developed using a new approach to construction of a discrete integration by parts formula, more precisely, the mimetic inner products [7,3,1]. Each mimetic inner product is a specially designed quadrature rule for volume integration that uses only degrees of freedom. This new approach removed the limitations of the SO method, increased significantly the class of admissible meshes, and allowed us to develop a convergence theory, e.g. [1,7].

The primary mimetic operators are coordinate-invariant; however, accuracy of the derived operators depends on accuracy of the inner products that have to be developed and analyzed for each curvilinear coordinate system. The number of publications dedicated to curvilinear coordinates is quite limited. First analysis in the cylindrical coordinates was done in [15]. The authors exploited flexibility of the mimetic framework to design a special inner product to preserve cylindrical symmetry on polar meshes [15]. In this paper, we consider equations of magnetic diffusion in cylindrical coordinates (r, z). Using a consistency argument as the main design principle, we build a new family of mimetic schemes that are second-order accurate (in the L^2-norm) on arbitrary polygonal meshes. Moreover, a consistent calculation of the Joule heating is intrinsic within each member of this family. More precisely, the average EM energy (e.g. see formula (8)) calculated in each polygonal cell is constant for every constant electric field $\mathbf{E}$.

Construction of an accurate mimetic inner product in the space of discrete electric fields is the key to the development of an accurate scheme. This construction is based on a discrete form of a local consistency condition. We show that the derivation of the discrete consistency condition needs to use the first-order quadratures for volume integral and the second-order quadratures for edge integrals.

The methodology discussed in this paper is added to a host Lagrangian-based Multiphysics code (FLAG). We test the implementation on cylindrical diffusion problems with either constant or spatially variant resistivities. We verify the implemented coupling between Ohmic heating and material temperature evolution. We also address the applicability of the method to liner implosion problems by comparison to an analytic finite-thickness slug model for magnetically-driven cylindrical liner implosions. The suitability of the implemented method to solve problems with heterogeneous conductivity distributions is also addressed.

The paper outline is as follows. In Section 2, we derive equations of magnetic diffusion. In Section 3, we develop a new mimetic finite difference method for solving magnetic diffusion in cylindrical coordinates. In Section 4, we perform convergence analysis of the developed scheme and explore the performance of the implemented method on a series of physical problems with cylindrical symmetry.

2. Problem formulation

Let us consider Maxwell's equations:

$$\mathrm{div}(\mathbf{D}) = \mathrm{div}(\varepsilon \mathbf{E}) = 0, \tag{1}$$

$$\mathrm{curl}(\mathbf{H}) = \frac{1}{\mu}\mathrm{curl}(\mathbf{B}) = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \tag{2}$$