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# High-order unconditionally stable FC-AD solvers for general smooth domains II. Elliptic, parabolic and hyperbolic PDEs; theoretical considerations

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ABSTRACT

A new PDE solver was introduced recently, in Part I of this two-paper sequence, on the basis of two main concepts: the well-known Alternating Direction Implicit (ADI) approach, on one hand, and a certain "Fourier Continuation" (FC) method for the resolution of the Gibbs phenomenon, on the other. Unlike previous alternating direction methods of order higher than one, which only deliver unconditional stability for rectangular domains, the new high-order FC-AD (Fourier-Continuation Alternating-Direction) algorithm yields *unconditional stability for general domains*—at an $\mathcal{O}(N\log(N))$ cost per time-step for an $N$ point spatial discretization grid. In the present contribution we provide an overall theoretical discussion of the FC-AD approach and we extend the FC-AD methodology to linear hyperbolic PDEs. In particular, we study the convergence properties of the newly introduced FC(Gram) Fourier Continuation method for both approximation of general functions and solution of the alternating-direction ODEs. We also present (for parabolic PDEs on general domains, and, thus, for our associated elliptic solvers) a stability criterion which, when satisfied, ensures unconditional stability of the FC-AD algorithm. Use of this criterion in conjunction with numerical evaluation of a series of singular values (of the alternating-direction discrete one-dimensional operators) suggests clearly that the fifth-order accurate class of parabolic and elliptic FC-AD solvers we propose is indeed unconditionally stable for all smooth spatial domains and for arbitrarily fine discretizations. To illustrate the FC-AD methodology in the hyperbolic PDE context, finally, we present an example concerning the Wave Equation—demonstrating sixth-order spatial and fourth-order temporal accuracy, as well as a *complete* absence of the debilitating "dispersion error", also known as "pollution error", that arises as finite-difference and finite-element solvers are applied to solution of wave propagation problems.



## 1. Introduction

The FC-AD (Fourier-Continuation Alternating-Direction) methodology introduced in [1] (Part I of this two-paper sequence) relies on two main elements: a novel spectral technique for general spatial domains (which is based on the one-dimensional Fourier Continuation method introduced in Part I) and the classical ADI approach pioneered by Douglas, Peaceman and Rachford [2–6]. Unlike previous alternating direction methods of order higher than one, which only deliver uncon-

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ditional stability for rectangular domains, the new high-order FC-AD algorithm yields *unconditional stability for general domains*—at an $\mathcal{O}(N\log(N))$ cost per time-step for an $N$ point spatial discretization grid. In the present contribution we provide an overall theoretical discussion of the FC-AD approach concerning unconditional stability and accuracy in the (linear) parabolic and elliptic contexts, and we extend the FC-AD methodology to problems concerning wave propagation and scattering. In conjunction with numerical evaluation of a series of singular values (of the alternating-direction discrete one-dimensional operators), our theory suggests clearly that the fifth-order accurate class of parabolic and elliptic FC-AD solvers we propose is indeed unconditionally stable for all smooth spatial domains and for arbitrarily fine discretizations. To illustrate the FC-AD methodology in the hyperbolic PDE context, finally, we present an example concerning the Wave Equation—demonstrating sixth-order spatial and fourth-order temporal accuracy, as well as *complete* absence of the debilitating "dispersion error", also known as "pollution error", that arises as finite-difference and finite-element solvers are applied to solution of wave propagation problems.

(A number of attempts have been made to combine the unconditional stability of the alternating direction type schemes with the spectral character of Fourier bases [7–10]. We expect that, like our FC-AD method, these Fourier-based approaches do not suffer from pollution errors. These previous efforts did not provide stable Fourier-based alternating-direction solvers for non-rectangular geometries; a more detailed discussion in these regards as well as comments concerning related spectral and spectral-element methodologies are given in the introduction to Part I.)

The appeal of the implicit alternating direction algorithms lies in the efficiency that results from their achievement of unconditional stability at a reduced cost per time-step. An important limitation has hindered the usefulness of the ADI, however: previous alternating direction methods could not be directly applied to PDEs on arbitrary (non-rectangular) domains without reducing the truncation error near the boundary to first order [11]. We note that while the ADI has been applied to problems on non-rectangular geometries [12–14], these applications were based on mappings of the PDE domains to rectangular regions—a procedure that is generally prohibitively laborious. To our knowledge, the FC-AD approach provides the first high-order accurate unconditionally stable alternating-direction scheme for general domains that does not rely on domain mappings.

A general discussion of current research on finite-difference and finite-element methods in the parabolic case for both simple and complex geometries was provided in Part I; here it is useful to summarize some of the main conclusions we have drawn as we placed the parabolic FC-AD algorithms in the context of the underlying literature. For diffusion equations the most notable advantage provided by the FC-AD approach lies in its unconditional stability for general domains: in Part I we demonstrated, for example, an improvement of a factor of 1000 in computing times, for engineering accuracies, over the computing time required by state of the art methodologies. Another interesting comparison concerns the contribution [15], which proposes a SAT method of order four of spatial and temporal accuracy for the diffusion equation: to our knowledge, this work introduces the SAT parabolic solver of highest demonstrated order of spatial accuracy. (Unlike the CFL condition for regular finite-difference methods, the SAT CFL restrictions are not affected as severely by small distances between the boundary and the nearest discretization points in the computational domain.) In view of their explicit character, however, existing SAT methods for parabolic equations do require time-steps proportional to the square of the spatial mesh-size, thus giving rise to high computing costs. In a direct comparison with the numerical example put forth in [15], for instance, our parabolic FC-AD solver produced the solution with accuracies matching the values $3 \times 10^{-4}$, $5 \times 10^{-5}$ and $1 \times 10^{-5}$ shown in Fig. 13 of that reference, in computational times that we estimate to be of the order of 80–100 times faster than those required by the method introduced in that reference. Such improvement factors result mainly for the fact that our unconditionally stable solver can produce the prescribed accuracies with a number of approximately 100 times fewer time-steps than the, e.g. 50,000 time-steps used by the SAT method in conjunction with its coarsest spatial discretization. These improvement-factor estimates take into account the slightly super-linear FFT cost and the cost arising from the fourth-order Richardson extrapolation inherent in our solver, as well as the cost arising from the fourth-order Runge–Kutta and nine-point finite differences stencil used in the method [15].

As mentioned above, besides an analysis of the parabolic and elliptic FC-AD solvers introduced previously, in this paper we put forward new FC-AD algorithms for the Wave Equation in two and three spatial dimensions. As is well known, spectral approaches provide major advantages over other methodologies for the solution of wave propagation problems. Indeed, owing to the accumulation of phase errors over multiple wave-cycles in long wave-trains, finite-difference and finite-element methods typically give rise to significant "dispersion errors", also known as "pollution errors", and thus require use of very large numbers of points per wavelength (PPW) in large-scale problems [16]. This difficulty was discussed in detail in [17,18] in the contexts of finite-difference and finite-element methods (FEM), respectively. It has long been recognized, further, that spectral methods generally do not suffer from this difficulty. As might be expected in view of the spectral nature of the FC-AD algorithms, the same is true of our Wave Equation FC-AD approach. Thus, the new FC-AD Wave Equation solver combines the low PPW-requirements typical of spectral solvers together with the geometric flexibility, high-order accuracy and unconditional stability otherwise inherent in the parabolic and elliptic FC-AD solvers.

To demonstrate the significant advantages offered by the (essentially dispersionless) FC method in the hyperbolic context we compare its performance with that resulting from finite-difference solvers of second- and fourth-orders of accuracy. In order to avoid difficulties associated with enforcement of boundary conditions in the finite-difference context, the finite-difference tests we perform involve periodic geometries only; our FC simulations, in turn, involve non-periodic, complex-geometry cases. The relevance of such comparisons becomes apparent when one considers that second- and fourth-order is indeed the state of the art accuracy-order for finite-difference solvers in complex domains: general-domain solvers recently made