Journal of Computational Physics 229 (2010) 3382–3401

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Linearity preserving nine-point schemes for diffusion equation on distorted quadrilateral meshes

Jiming Wu *, Zihuan Dai, Zhiming Gao, Guangwei Yuan

Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing 100088, PR China

ARTICLE INFO

Article history:
Received 30 March 2009
Received in revised form 9 December 2009
Accepted 6 January 2010
Available online 15 January 2010

Keywords:
Diffusion equation
Difference scheme
Linearity preserving method
Distorted mesh

ABSTRACT

In this paper, we employ the so-called linearity preserving method, which requires that a difference scheme should be exact on linear solutions, to derive a nine-point difference scheme for the numerical solution of diffusion equation on the structured quadrilateral meshes. This scheme uses firstly both cell-centered unknowns and vertex unknowns, and then the vertex unknowns are treated as a linear combination of the surrounding cell-centered unknowns, which reduces the scheme to a cell-centered one. The weights in the linear combination are derived through the linearity preserving approach and can be obtained by solving a local linear system whose solvability is rigorously discussed. Moreover, the relations between our linearity preserving scheme and some existing schemes are also discussed, by which a generalized multipoint flux approximation scheme based on the linearity preserving criterion is suggested. Numerical experiments show that the linearity preserving schemes in this paper have nearly second order accuracy on many highly skewed and highly distorted structured quadrilateral meshes.

1. Introduction

In many computational problems, such as the computational radiation hydrodynamics and the reservoir simulation, the discretization of the following diffusion term

$$\nabla \cdot (\kappa(x,y)\nabla u) \tag{1.1}$$

is of great interests. In the radiation hydrodynamics, u can be a certain material temperature and $\kappa(x,y)$ is the diffusion coefficient, while in reservoir simulation, u and $\kappa(x,y)$ denote the pressure and permeability, respectively.

In discretizing (1.1), one usually has to face two difficulties, i.e., (i) $\kappa(x,y)$ is discontinuous and strongly nonlinear, (ii) the mesh is highly distorted and highly skewed which usually occurs in the Lagrangian or ALE hydrodynamic computations [8,10]. There has been extensive study on developing efficient numerical schemes for (1.1), the issues about which range from the classical stability and accuracy to some other desirable numerical properties, including symmetry and positive definiteness of the resulting linear system, local stencil, local conservation, positivity preserving or monotonicity, simplicity, robustness, cost-efficiency, etc. To our knowledge, there exists no scheme satisfying all the above properties. Usually, a scheme possesses some properties at the cost of losing other ones. Among the aforementioned desirable properties, from our point of view, the accuracy and stability are the fundamental ones.

In this paper, we are more interested in the so-called linearity preserving property, which says that a difference scheme is exact on linear solutions. We observe that some authors mentioned this property in their works [6,7,22,23,27], for example,

* Corresponding author.
E-mail addresses: wu_jiming@iapcm.ac.cn (J. Wu), dai_zihuan@iapcm.ac.cn (Z. Dai), gao@iapcm.ac.cn (Z. Gao), yuan_guangwei@iapcm.ac.cn (G. Yuan).

0021-9991/$ - see front matter
doi:10.1016/j.jcp.2010.01.007

the authors in [22] suggested a so-called MDHW scheme, which preserves the homogeneous linear solutions $a + bx + cy$ and $a + br + cz$ in $x - y$ and $r - z$ geometries, respectively. To our knowledge, the authors usually claimed that certain scheme has or has not the linearity preserving property, but there has been no intensive investigation or study on this topic. Although the linearity preserving property has not been proved theoretically to be a sufficient or necessary condition for certain good numerical properties mentioned above, we observe from our numerical practice and other peoples' works that a difference scheme with linearity preserving property usually has good accuracy on highly distorted meshes. Motivated by this observation, we suggest here the so-called *linearity preserving criterion*, which requires that each step of the derivation of a difference scheme for diffusion equation is exact or linearly exact, i.e., exact in the sense whenever the solution is a linear function and the diffusion coefficient $\kappa(x, y)$ is a constant. Obviously, a difference scheme derived from the linearity preserving criterion is exact for the linear solution.

The linearity preserving criterion is applied here to improve the accuracy of a special difference scheme suggested originally in [20] through a control volume approach. In this scheme, the normal component of the flux (see (2.5)) on each cell edge is explicitly expressed by the two cell-centered unknowns with respect to the cells sharing that edge, and the two vertex unknowns defined at the two endpoints of the edge. Usually the vertex unknowns are treated as intermediate ones and are expressed by a linear combination of the surrounding cell-centered unknowns. On structured quadrilateral meshes, the above scheme involves nine cell-centered unknowns and as a result, is often called as the nine-point scheme (NPS) [14,28]. We note that Klausen and Winther [19] once gave a definition of the multipoint flux approximation (MPFA), which states that the MPFA discretization is a control volume method where more than two pressure values (here the values of u) are used to give an explicit discrete flux expression. According to this recent definition of MPFA, NPS can also be viewed as, to some sense, a kind of MPFA scheme.

The most important features of NPS are that it has a very simple explicit expression of the flux, involves less amount of computational cost and is easy for coding, so that NPS has been used for a long time in some hydrodynamics codes, such as LARED-I and MARED [11,26]. The main disadvantage of NPS is that it loses accuracy on highly distorted meshes, which is caused mainly by the rough or improper treatments on the vertex unknowns. How to improve the accuracy of NPS is a very interesting problem and has drawn some authors' attentions [5,7,28,30,33]. These improvements are either complicated and costly or not accurate enough. It is evident that a desirable improvement on NPS should keep its main advantages so that it is simple for coding and involves less amount of computational cost and moreover, does not result in a major change of the original codes.

We use the linearity preserving criterion to rederive the NPS and further, to obtain a simple treatment for the vertex unknowns. The computational cost of this new vertex unknown treatment is approximately one third of that in [28] and increases the accuracy to almost second order on many typical highly distorted and highly skewed meshes. As done in [28], our improvement treats the discontinuity rigorously and furthermore, is obtained not at the cost of massive change of the original codes. Since we design our algorithm by using the continuity of the flux, the scheme derived here also keeps the local conservation. For a treatment of vertex unknown, both the method in [28] and our present algorithm here depend upon the solution of a local 4×4 linear system, whose unknowns in the former are the weights in the linear combination mentioned above while in the present paper are some newly introduced ones. Compared with [28], the present local linear system has a simple structure and a simple explicit expression for its entries, which reduces largely the computational cost and makes it possible to analyze solvability.

In using the linearity preserving method to find a treatment for the vertex unknowns, we employ an MPFA-type technique to introduce some intermediate cell edge unknowns. The only difference is that our cell edge unknown is defined at a dynamic point on the whole edge, instead of a fixed point (known as continuity point) in certain half edge, such as the midpoint of a cell edge used in the usual MPFA type schemes. The idea to choose a dynamic point on the whole cell edge enables us not only to obtain a robust algorithm for the vertex unknowns but also to construct a generalized MPFA scheme.

In the construction of many discretization schemes for the diffusion equation, such as the local support operator scheme (LSOM) [23], the local flux mimetic finite difference scheme (LFMFD) [21], the physical space derived MPFA [1,19] together with its variations [9], the reference space derived MPFA [3,29], and the nine-point scheme in [28], one has to solve certain local linear systems. The local systems in some schemes, such as LSOM and the symmetric version of LFMFD, are symmetric and positive definite for meshes consisting of convex cells and as a consequence, the solvability of the local systems follows immediately. However, the solvability of the local linear systems in other schemes is seldom discussed and the corresponding algorithms run the risk of being breakdown in the computational course. This problem is neglected mainly because the breakdown rarely occurs in practical computation, however, theoretically speaking, it does exist. To our practice, the possibility for the breakdown increases when the cells in the mesh approach concave ones. We note that the authors in [19] obtained the solvability of the local linear system in a special MPFA scheme under the condition that the symmetric part of certain 2×2 matrix is positive definite. Usually, this condition is not satisfied by many highly distorted meshes. By introducing the dynamic continuity point, we are able to discuss rigorously the solvability of our local linear system. Then, the difficulty that arises from the possible singularity of our local linear system is overcome, which makes our algorithm a robust one. More interesting is that the discussion for our local linear system also contributes to the MPFA algorithm, since we notice that there exists certain relation between our linearity preserving nine-point scheme and the physical space derived MPFA.

The rest of this paper is organized as follows. In Section 2, we derive the NPS scheme by the linearity preserving method. In Section 3, we discuss in details the treatments for the vertex unknowns and in Section 4, we give the relations between our