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# Well-balanced and energy stable schemes for the shallow water equations with discontinuous topography

Ulrik S. Fjordholm [a], Siddhartha Mishra [b], Eitan Tadmor [c,*]

[a] Seminar for Applied Mathematics (SAM), Department of Mathematics, ETH Zürich, HG J 48, Zürich 8092, Switzerland
[b] Seminar for Applied Mathematics (SAM), Department of Mathematics, ETH Zürich, HG G 57.2, Zürich 8092, Switzerland
[c] Department of Mathematics, Center of Scientific Computation and Mathematical Modeling (CSCAMM), Institute for Physical Sciences and Technology (IPST), University of Maryland, MD 20742-4015, USA

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ABSTRACT

We consider the shallow water equations with non-flat bottom topography. The smooth solutions of these equations are energy conservative, whereas weak solutions are energy stable. The equations possess interesting steady states of lake at rest as well as moving equilibrium states. We design energy conservative finite volume schemes which preserve (i) the lake at rest steady state in both one and two space dimensions, and (ii) one-dimensional moving equilibrium states. Suitable energy stable numerical diffusion operators, based on energy and equilibrium variables, are designed to preserve these two types of steady states. Several numerical experiments illustrating the robustness of the energy preserving and energy stable well-balanced schemes are presented.



## 1. Introduction

Flows in lakes, rivers, irrigation channels and near-shore oceanic flows are of great interest in hydrology, oceanography and climate modeling. Common to all of these flows is the fact that vertical scales of motion are much smaller than the horizontal scales. By this and the assumption of hydrostatic balance (see [41]), the incompressible Navier–Stokes equations of fluid dynamics can be simplified and reduce to the so-called shallow water equations

$$\begin{aligned} &h_t + (hu)_x + (hv)_y = 0, \\ &(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y = -ghb_x, \\ &(hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y = -ghb_y. \end{aligned} \tag{1.1}$$

Here, $h$ is the height of the fluid column and $(u, v)$ is the velocity field. The constant $g$ is the acceleration due to gravity and the function $b \equiv b(x, y)$ represents the bottom topography of the surface over which the fluid flows. In general, the bottom topography can be rather complicated and possibly discontinuous. We have neglected eddy viscosity in the above equation. When the variation of the unknowns in the $y$-direction are negligible, one may find the one-dimensional version of (1.1) by setting $v$ and all the derivatives in the $y$-direction to zero, thus obtaining the system

* Corresponding author. Tel.: +1 301 405 0648; fax: +1 301 314 6673.
*E-mail addresses:* ulrikf@sam.math.ethz.ch (U.S. Fjordholm), smishra@sam.math.ethz.ch (S. Mishra), tadmor@cscamm.umd.edu (E. Tadmor).

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$$\begin{aligned} &h_t + (hu)_x = 0, \\ &(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = -ghb_x. \end{aligned} \tag{1.2}$$

The shallow water system with topography (1.1) amounts to a system of *balance laws*,

$$U_t + f(U)_x + g(U)_y = -s(x,y,U), \tag{1.3}$$

where $U = [h,hu,hv]^\top$ is the vector of unknowns, $f = \left[hu, hu^2 + \frac{1}{2}gh^2, huv\right]^\top$ and $g = \left[hv, huv, hv^2 + \frac{1}{2}gh^2\right]^\top$ are the flux vectors, and $s = [0,ghb_x,ghb_y]^\top$ is the source vector.

If the bottom topography is flat, i.e. $b \equiv Const.$, then (1.1) is reduced to the standard shallow water equations without topography, which is a strictly hyperbolic system of conservation laws,

$$U_t + f(U)_x + g(U)_y = 0. \tag{1.4}$$

It is well-known that solutions of the conservation law (1.4), and likewise, solutions of the balance law (1.3), can develop shock discontinuities in a finite time, independent of whether the initial data is smooth or not. Hence, the solutions of balance laws (1.3) are considered in the weak sense and are well-defined as long as the source $s$ remains uniformly bounded [7]. In particular, weak solutions of (1.1) are well-defined under the assumption that the topography function $b$ is in $W^{1,\infty}(\mathbb{R}^2)$. However, difficulties arise when the topography function is discontinuous: the action of the source term on the right of (1.1) can be interpreted as a non-conservative product (see [8]), or by a limiting smoothing process of $b$.

### 1.1. The entropy condition

Weak solutions of conservation laws (1.4), and likewise, weak solutions of the balance law (1.3), need not be unique. Another aspect of non-uniqueness enters (1.1) through the action of the source term $s(x,y,U) = -gh\nabla b(x,y)$: its interpretation as a non-conservative product or using a limiting smoothing process depends on a non-unique choice of a path integral. To address this issue of non-uniqueness, an additional admissibility criterion is imposed, based on the so-called *entropy condition*. To this end, one assumes that the general system of balance laws (1.3) is equipped with a convex entropy function $E = E(U)$, associated entropy flux functions $H = H(U)$, $K = K(U)$ and $J = [J_1(x,y,U),J_2(x,y,U)]^\top$, such that the following compatibility relations, expressed in terms of the vector of *entropy variables* $V := \partial_U E$, hold:

$$\partial_U H = \langle V, \partial_U f(U)\rangle, \quad \partial_U K = \langle V, \partial_U g(U)\rangle, \quad \partial_x J_1 + \partial_y J_2 = \langle V, s\rangle. \tag{1.5a}$$

Multiplying (1.3) by $V = \partial_U E$, the compatibility relations (1.5a) imply that smooth solutions of (1.4) satisfy the conservation law

$$E(U)_t + (H(U) + J_1)_x + (K(U) + J_2)_y = 0. \tag{1.5b}$$

Conversely, if this additional conservation law holds for *all smooth* functions $U$, then $E$ is an entropy function, i.e., (1.5a) holds with the entropy fluxes $H$, $K$ and $J$. This balance between the entropy and entropy fluxes has to be modified to take into account the presence of possible discontinuities in (1.3): we postulate that the discontinuous solution $U$ of the balance laws (1.3) can be realized by a vanishing viscosity limit, which in turn leads to the distributional entropy inequality

$$E(U)_t + (H(U) + J_1)_x + (K(U) + J_2)_y \leqslant 0. \tag{1.5c}$$

In the absence of a source term ($s \equiv 0$), (1.5c) amounts to the usual entropy condition for conservation laws [7]. Scalar conservation laws are equipped with infinitely many entropy pairs – indeed, every convex function serves as a scalar entropy function, and this paves the way for a proof of existence, uniqueness and stability in the scalar framework. For general systems of conservation laws, however, the existence of entropy pairs places a compatibility restriction on the structure of the fluxes $f(\cdot)$ and $g(\cdot)$ which is not always met. Similarly, general systems of balance laws need not possess entropy functions, except for special systems which are endowed with at least one entropy function. Observe that in the particular case of balance laws, the source term, $s$ also has to have a special structure for the entropy compatibility (1.5a) to hold.

An illustrative example is provided by the shallow water system with bottom topography (1.1). Here, the total energy

$$E(U) := \frac{1}{2}(hu^2 + hv^2 + gh^2 + ghb)$$

serves as an entropy function. The total energy $E(U)$ consists of the kinetic energy $h(u^2 + v^2)/2$ and the gravitational potential energy $gh(h + b)$, which involves the bottom topography $b$. A straightforward calculation reveals that if $U$ is a smooth solution of (1.1) then

$$E(U)_t + \left(\frac{1}{2}(hu^3 + huv^2) + ghu(h+b)\right)_x + \left(\frac{1}{2}(hu^2v + hv^3) + ghv(h+b)\right)_y = 0. \tag{1.6}$$

Thus, $E(U)$ is an entropy function associated with entropy fluxes

$$H(U) := \frac{1}{2}(hu^3 + huv^2) + gh^2u, \quad K(U) := \frac{1}{2}(hu^2v + hv^3) + gh^2v, \quad J := ghb[u, v]^\top.$$