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Improvement of the recursive projection method for linear iterative scheme stabilization based on an approximate eigenvalue problem

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ABSTRACT

An algorithm for stabilizing linear iterative schemes is developed in this study. The recursive projection method is applied in order to stabilize divergent numerical algorithms. A criterion for selecting the divergent subspace of the iteration matrix with an approximate eigenvalue problem is introduced. The performance of the present algorithm is investigated in terms of storage requirements and CPU costs and is compared to the original Krylov criterion. Theoretical results on the divergent subspace selection accuracy are established. The method is then applied to the resolution of the linear advection–diffusion equation and to a sensitivity analysis for a turbulent transonic flow in the context of aerodynamic shape optimization. Numerical experiments demonstrate better robustness and faster convergence properties of the stabilization algorithm with the new criterion based on the approximate eigenvalue problem. This criterion requires only slight additional operations and memory which vanish in the limit of large linear systems.

1. Introduction

Iterative methods for linear systems have taken a dominant role in the computation of problems in fluid dynamics and aerodynamics. This is the case of implicit time discretizations for problems with a high number of degrees of freedom (DOFs) which require the inversion of a large matrix for which direct solvers often become ineffective. However, these techniques can suffer from some limited asymptotic convergence properties and can lead to restrictions on the time step for practical applications. Preconditioning methods become necessary but are not always sufficient. Artificial dissipation methods may also be used to ensure stability and enhance robustness [1] but one must be careful not to affect the accuracy and dissipation properties of the numerical scheme. The recursive projection method (RPM) introduced by Shroff and Keller [2] constitutes an alternative technique. The RPM was initially developed for extending the domain of convergence of fixed-point iterative procedures in the context of bifurcation analysis [2–4]. This method aims at identifying the diverging eigenmodes of the iteration matrix of the numerical scheme. Then, the method applies Newton iterations in the subspace spanned by the associated eigenvectors while keeping the original scheme in its orthogonal complement. The RPM has also been applied either to stabilize iterative procedures or to accelerate convergence to steady-state solutions [5–7].

The RPM is however sometimes inefficient in the case of large linear systems due to the existence of modes with large negative real parts that reduce the asymptotic convergence rate of the RPM algorithm. Davidson [3] and Janovskỳ and Liberda [8] improved the performance of the RPM in the context of continuation of invariant subspaces. They used a preconditioner based on a Cayley transform of the Jacobian matrix in order to modify the mapping of the eigenvalues thus eliminating the influence of these modes. However the Cayley transform requires the inversion of a matrix of same size as the Jacobian

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matrix and is therefore unsuitable for the resolution of linear systems which is the focus of this study. Moreover, the RPM needs additional storage space and operation counts compared to the original algorithm which is the reason why this method is mainly used for algorithm stabilization rather than for convergence acceleration. The approach introduced in this study improves the RPM performances by reducing memory requirements and CPU time.

Numerical methods for gradient computation in the context of aerodynamic shape optimization constitute a typical example of robustness issue for practical applications. Sensitivity analysis for aerodynamic shape optimization has recently become of increased importance for a variety of applications [9]. It requires the computation of gradients of functionals in the design parameter space. Among sensitivity evaluation methods, the discrete methods require the solution of a linear system that results from the differentiation of the discrete equations of the nonlinear problem. For problems involving turbulent flows solved using the Reynolds-Averaged Navier–Stokes (RANS) equations, there are many situations where the numerical method for the resolution of the linear problem diverges even if it is not the case for the nonlinear problem [10–14]. If a method converges asymptotically for the nonlinear problem, it will also do so for the linear problem. The issue is that methods that do not converge asymptotically in the nonlinear case often stall at some small residual, and are therefore still useful. The same methods applied to the linear problem diverge. For instance, Dwight and Brezillon [15] have studied the differentiation of the Spalart–Allmaras turbulence model for the discrete adjoint method. Numerical results show that different approximations in the Spalart–Allmaras differentiation could lead to a poorly conditioned linear system and a divergent algorithm. Nemec and Zingg [16] successfully applied an incomplete lower-upper preconditioned GMRES to the discrete adjoint method for the two-dimensional (2D) RANS equations with the Spalart–Allmaras turbulence model. However, memory requirements and loss of preconditioner efficiency make this technique inappropriate to large stiff problems [16,15].

For situations in which the iterative method fails, the application of RPM constitutes a robust algorithm for solving sensitivity analysis problems. The resolution of either the adjoint compressible RANS equations coupled with the one-equation turbulence model of Spalart–Allmaras [13], or the direct compressible RANS equations coupled with the two-equation model of Launder–Sharma [12] were successfully stabilized using this method. In the following, we will consider the numerical methods introduced in Ref. [12] as a model to assess the developments on the RPM.

The purpose of the present study is to introduce a new procedure for enhancing the robustness and performance of the RPM. The method is based on a new criterion for selecting the divergent subspace with an approximate eigenvalue problem (AEP) of the iteration matrix, called the AEP criterion. The paper is organized as follows. Section 2 presents the stabilization procedure. The RPM is described in Section 2.1. The original Krylov criterion and the AEP criterion for selecting the divergent subspace are presented in Sections 2.2 and 2.3, respectively. Theoretical results and algorithm analyses are also shown. Numerical results are presented for the linear advection–diffusion equation in Section 3.1 together with a sensitivity analysis of a turbulent transonic flow over a bump in Section 3.2. Finally, conclusions are summarized in Section 4.

2. Recursive projection method

2.1. Stabilization procedure

We are concerned with solutions $\mathbf{x}$ of linear systems

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{N\times N}$ is a square matrix and $\mathbf{b} \in \mathbb{R}^N$ is the right-hand side. Many iterative procedures for solving this problem belong to the family of fixed-point iterations that have the form

$$\mathbf{x}^{(l+1)} = \mathbf{F}(\mathbf{x}^{(l)}) = \mathbf{\Phi}\mathbf{x}^{(l)} + \mathbf{M}^{-1}\mathbf{b}, \tag{2}$$

where $\mathbf{\Phi} = \mathbf{I} - \mathbf{M}^{-1}\mathbf{A}$ represents the iteration matrix of the numerical scheme and $\mathbf{M}$ is a preconditioning matrix.

Suppose $\mathbf{A}$ and $\mathbf{M}$ are nonsingular, then whether or not the iteration (2) converges to the solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ depends upon the eigenvalues of $\mathbf{\Phi}$ (see for instance Ref. [17]). Here, we follow the RPM introduced by Shroff and Keller [2] for the stabilization of unstable recursive fixed-point procedures. Suppose the iteration (2) diverges thus implying that there are $m \geqslant 1$ eigenvalues of $\mathbf{\Phi}$ with modulus greater than unity:

$$|\lambda_1| \geqslant \cdots \geqslant |\lambda_m| \geqslant 1. \tag{3}$$

These eigenvalues are called the divergent eigenvalues. The eigenvalue with greater modulus is commonly referred to as the dominant eigenvalue. Define the divergent subspace

$$\mathbb{P} = \mathrm{span}\{\mathbf{e_1}, \ldots, \mathbf{e_m}\} \tag{4}$$

spanned by the eigenvectors associated with eigenvalues (3), and $\mathbb{Q} = \mathbb{P}^\perp$ its orthogonal complement in $\mathbb{R}^N$. These subspaces form a direct sum of $\mathbb{R}^N$, therefore every vector can be decomposed in an unique way as the sum

$$\forall \mathbf{x} \in \mathbb{R}^N, \quad \exists(\mathbf{x}_p, \mathbf{x}_q) \in \mathbb{P} \times \mathbb{Q} : \quad \mathbf{x} = \mathbf{x}_p + \mathbf{x}_q. \tag{5}$$

The orthogonal projectors onto the subspaces $\mathbb{P}$ and $\mathbb{Q}$ are denoted $\mathbf{P}$ and $\mathbf{Q}$, respectively. These projectors may be defined from an orthonormal basis $\mathbf{V} \in \mathbb{R}^{N\times m}$ for $\mathbb{P}$ in the following way

$$\mathbf{P} = \mathbf{V}\mathbf{V}^\top, \quad \mathbf{Q} = \mathbf{I} - \mathbf{V}\mathbf{V}^\top, \tag{6}$$