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A comparison of companion matrix methods to find roots of a trigonometric polynomial

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ABSTRACT

A trigonometric polynomial is a truncated Fourier series of the form $f_N(t) \equiv \sum_{j=0}^{N} a_j \cos(jt) + \sum_{j=1}^{N} b_j \sin(jt)$. It has been previously shown by the author that zeros of such a polynomial can be computed as the eigenvalues of a companion matrix with elements which are complex valued combinations of the Fourier coefficients, the "CCM" method. However, previous work provided no examples, so one goal of this new work is to experimentally test the CCM method. A second goal is introduce a new alternative, the elimination/Chebyshev algorithm, and experimentally compare it with the CCM scheme. The elimination/Chebyshev matrix (ECM) algorithm yields a companion matrix with real-valued elements, albeit at the price of usefulness only for real roots. The new elimination scheme first converts the trigonometric rootfinding problem to a pair of polynomial equations in the variables (c, s) where $c \equiv \cos(t)$ and $s \equiv \sin(t)$. The elimination method next reduces the system to a single univariate polynomial $P(c)$. We show that this same polynomial is the resultant of the system and is also a generator of the Groebner basis with lexicographic ordering for the system.

Both methods give very high numerical accuracy for real-valued roots, typically at least 11 decimal places in Matlab/IEEE 754 16 digit floating point arithmetic. The CCM algorithm is typically one or two decimal places more accurate, though these differences disappear if the roots are "Newton-polished" by a single Newton's iteration. The complex-valued matrix is accurate for complex-valued roots, too, though accuracy decreases with the magnitude of the imaginary part of the root. The cost of both methods scales as $O(N^3)$ floating point operations. In spite of intimate connections of the elimination/Chebyshev scheme to two well-established technologies for solving systems of equations, resultants and Groebner bases, and the advantages of using only real-valued arithmetic to obtain a companion matrix with real-valued elements, the ECM algorithm is noticeably inferior to the complex-valued companion matrix in simplicity, ease of programming, and accuracy.

1. Introduction

When a Fourier series is truncated to its first $(2N + 1)$ terms, the result is a generalized polynomial:

Definition 1 (*Trigonometric Polynomial*). A truncated Fourier series of the form

$$f_N(t) \equiv \sum_{j=0}^{N} a_j \cos(jt) + \sum_{j=1}^{N} b_j \sin(jt) \tag{1}$$

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is a trigonometric polynomial of degree N, which contains a total of $2N+1$ terms.

A trigonometric polynomial $f_N(t)$ has exactly $2N$ roots, when the roots are counted according to their multiplicity, in the strip of the complex plane such that $-\pi < \Re(t) \leqslant \pi$ as proved in [10]. (Note that roots with $\Re(t) = -\pi$ are excluded from this count.) If the coefficients for $f_N(t)$ are real, then the number of real roots is always even, again counting roots according to their multiplicity so that a single double root counts as two [10].

A Fourier companion matrix method with complex-valued elements was proposed by the author in the review article [10]. However, no numerical illustrations or practical tests were provided. Is the "CCM" algorithm accurate in finite precision arithmetic? One goal here is to provide numerical experiments to answer this question.

Here, we also propose a new algorithm ("ECM method") to compute the roots of a trigonometric polynomial. This yields a matrix whose elements are real-valued if the coefficients of the polynomial are real-valued. We also compare the two root-finding schemes through both theoretical and experimental means. *Both companion matrix methods are exact in the absence of roundoff error*, but floating point errors will be shown to favor the CCM method over the ECM procedure.

If the trigonometric polynomial contains *only* cosine terms or *only* sine terms, the roots are found much more efficiently by the parity-exploiting algorithms of [11]. We therefore exclude such pure cosine and pure sine polynomials from discussion here.

There is a wide variety of alternatives for computing trigonometric polynomial zeros, some restricted to truncated Fourier series, others applicable to almost any function $f(x)$. Doing a careful comparison would require a book, not merely research article. However, a discussion of Fourier-specific algorithms is given in Section 7 of [10] and the original sources [16,26,30,29,35,23].

2. A Fourier companion matrix with complex-valued elements: the CCM algorithm

The transformation

$$z = \exp(it) \tag{2}$$

converts a *trigonometric* polynomial $f_N(t)$ with $2N+1$ *terms into an ordinary* polynomial $h(z)$ of degree $2N$ as independently discovered several times [35,7,3]. The "associated polynomial", $h(z[t]) \equiv \exp(iNt) f_N(t)$, is

$$h(z) = \frac{1}{2}\sum_{k=0}^{2N} h_k z^k \equiv z^N f(t[z]) \tag{3}$$

where

$$h_j = \begin{cases} a_{N-j} + i\,b_{N-j}, & j = 0, 1, \ldots, (N-1) \\ 2\,a_0, & j = N \\ a_{j-N} - i\,b_{j-N}, & j = N+1, N+2, \ldots (2N) \end{cases} \tag{4}$$

From this transformation comes the following.

Theorem 1 (*Fourier Companion Matrix*). *Define the trigonometric polynomial*

$$f_N(t) \equiv \sum_{j=0}^{N} a_j \cos(jt) + \sum_{j=1}^{N} b_j \sin(jt) \tag{5}$$

The matrix elements B_{jk} of the Frobenius matrix for a trigonometric polynomial of general degree N (and therefore $(2N+1)$ terms) are

$$B_{jk} = \begin{cases} \delta_{j,k-1}, & j = 1, 2, \ldots, (2N-1) \\ (-1)\frac{h_{k-1}}{a_N - i b_N}, & j = 2N \end{cases} \tag{6}$$

where δ_{jk} is the usual Kronecker delta function such that $\delta_{jk} = 0$ if $j \neq k$ while $\delta_{jj} = 1$ for all j and k and the h_j are defined by (4).

The roots t_k of $f_N(t)$ are the negative of $\sqrt{-1}$ times the logarithm of the matrix eigenvalues z_k:

$$t_{k,m} \equiv \arg(z_k) + 2\pi m - i\log(|z_k|), \quad k = 1, 2, \ldots, 2N; \; m = \text{integer} \tag{7}$$

In particular, the real-valued roots of $f_N(t)$ for real $t \in [-\pi, \pi]$ are the angles of the roots of $h(z)$ on the unit circle. Equivalently, each real-valued root t_k of $f(t)$ on $t \in (-\pi, \pi]$ is connected to a root z_k of the associated polynomial through $t_k = \arg(z_k) \forall k$ such that $|z_k| = 1$. Here $\arg(z)$ is the usual complex argument function such that, for $z = |z|\exp(i\theta), \arg(z) = \theta$. From [10].

For $N = 2$, the Fourier–Frobenius matrix is explicitly

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (-1)\frac{a_2 + ib_2}{a_2 - ib_2} & (-1)\frac{a_1 + ib_1}{a_2 - ib_2} & (-1)\frac{2\,a_0}{a_2 - ib_2} & (-1)\frac{a_1 - ib_1}{a_2 - ib_2} \end{vmatrix} \tag{8}$$