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Constructing reference metrics on multicube representations of arbitrary manifolds

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ABSTRACT

Reference metrics are used to define the differential structure on multicube representations of manifolds, i.e., they provide a simple and practical way to define what it means globally for tensor fields and their derivatives to be continuous. This paper introduces a general procedure for constructing reference metrics automatically on multicube representations of manifolds with arbitrary topologies. The method is tested here by constructing reference metrics for compact, orientable two-dimensional manifolds with genera between zero and five. These metrics are shown to satisfy the Gauss–Bonnet identity numerically to the level of truncation error (which converges toward zero as the numerical resolution is increased). These reference metrics can be made smoother and more uniform by evolving them with Ricci flow. This smoothing procedure is tested on the two-dimensional reference metrics constructed here. These smoothing evolutions (using volume-normalized Ricci flow with DeTurck gauge fixing) are all shown to produce reference metrics with constant scalar curvatures (at the level of numerical truncation error).

1. Introduction

The problem of developing methods for solving partial differential equations numerically on manifolds with nontrivial topologies has been studied in recent years by a number of researchers. The most widely studied approach, the surface finite element method, was developed originally by Gerhard Dziuk and collaborators [1–5]. This method can be applied to manifolds having isometric embeddings as codimension one surfaces in $\mathbb{R}^n$. Triangular (or higher dimensional simplex) meshes on these surfaces are used to define discrete differential operators using fairly standard finite element methods. The topological structures of these manifolds are encoded in the simplicial meshes, while their differential structures and geometries are inherited by projection from the enveloping Euclidean $\mathbb{R}^n$. The surface finite element method has been used in a number of applications on surfaces, including various evolving surface problems [6,7] and harmonic map flows on surfaces with nontrivial topologies [8–10]. The method is somewhat restrictive in that it only applies to manifolds that can be embedded isometrically as codimension one surfaces in $\mathbb{R}^n$.

The surface finite element method has been generalized in different ways to allow the possibility of studying problems on larger classes of manifolds, which need not be embedded surfaces in $\mathbb{R}^n$. For instance, Michael Holst and collabora-

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tors [11–13] have developed methods for defining discrete representations of differential forms on simplicial representations of manifolds with arbitrary topologies. The differential structure of a manifold in this approach is determined by explicitly specifying the set of coordinate overlap maps that cover the interfaces between neighboring simplices. The geometry of the manifold (needed for example to define the covariant Laplace–Beltrami operator, or the dual transformations of differential forms) is determined in this approach by a metric on the manifold that must also be explicitly supplied. Oliver Sander and collaborators [14–18] have introduced a different generalization of the surface finite element method. Their approach, called the geodesic finite element method, uses the geometry of the manifold (which must be specified explicitly) to construct discrete differential operators that conform more precisely to the manifold. The usual interpolation rule along straight coordinate lines in the reference element is replaced with geodesic interpolation in a curved manifold. The global topology and the differentiable structures must be specified explicitly for each manifold. These approaches are very general, but they are somewhat cumbersome to use in practice since they require a great deal of detailed information to be explicitly provided in order to determine the differential and geometrical structures for each manifold studied.

Multicube representations of manifolds [19] provide a framework for the development of simpler methods for solving PDEs numerically on manifolds with arbitrary topologies. This approach, which we review in the following paragraphs, has several significant advantages over the finite element methods discussed above. For one, the multicube method represents a manifold as a mesh of non-overlapping cubes (or hypercubes) rather than simplices. This makes it simpler to introduce natural bases for vector and tensor fields on these manifolds. The cubic structure is also better suited for spectral numerical methods, which converge significantly faster than finite element methods of any (fixed) order. Another distinct advantage of the multicube approach is that the differential structures on multicube manifolds can be determined by a smooth reference metric. Therefore one need not specify the differential structure explicitly as would be required by the earlier generalizations of the surface finite element method. In our previous work involving the multicube method we specified the needed reference metrics analytically for the few simple manifolds that we studied [19,20]. In more complicated cases, however, the problem of finding an appropriate smooth reference metric is more difficult. The main purpose of this paper is to develop methods for generating the needed reference metrics automatically.

The multicube representation of a manifold Σ consists of a collection of non-intersecting n-dimensional cubic regions $\mathcal{B}_A \subset \mathbb{R}^n$ for $A = 1, 2, \ldots, N_R$, together with a set of one-to-one invertible maps $\Psi^{A\alpha}_{B\beta}$ that determine how the boundaries of these regions are to be connected together. The maps $\partial_\alpha \mathcal{B}_A = \Psi^{A\alpha}_{B\beta}(\partial_\beta \mathcal{B}_B)$ define these connections by identifying points on the boundary face $\partial_\beta \mathcal{B}_B$ of region $\mathcal{B}_B$ with points on the boundary face $\partial_\alpha \mathcal{B}_A$ of region $\mathcal{B}_A$ (cf. Ref. [19] and Appendix B). It is convenient to choose all these cubic regions in $\mathbb{R}^n$ to have the same coordinate size L, the same orientation, and to locate them so that regions intersect (if at all) in $\mathbb{R}^n$ only at faces that are identified by the $\Psi^{A\alpha}_{B\beta}$ maps. Since the regions do not overlap, the global Cartesian coordinates of $\mathbb{R}^n$ can be used to identify points in Σ. Tensor fields on Σ can be represented by their components in the tensor bases associated with these global Cartesian coordinates.

The Cartesian components of smooth tensor fields on a multicube manifold are smooth functions of the global Cartesian coordinates within each region $\mathcal{B}_A$, but these components may not be smooth (or even continuous) across the interface boundaries $\partial_\alpha \mathcal{B}_A$ between regions. Smooth tensor fields must instead satisfy more complicated interface continuity conditions defined by certain Jacobians, $J^{A\alpha i}_{B\beta j}$, that determine how vectors v^i and covectors w_i transform across interface boundaries: $v^i_A = J^{A\alpha i}_{B\beta j}\, v^j_B$ and $w_{Ai} = J^{*B\beta j}_{A\alpha i}\, w_{Bj}$. As discussed in Ref. [19], the needed Jacobians are easy to construct given a smooth, positive-definite reference metric $\tilde{g}_{ij}$ on Σ.

A smooth reference metric also makes it possible to define what it means for tensor fields to be C^1, i.e., to have continuous derivatives across interface boundaries. Tensors are C^1 if their covariant gradients (defined with respect to the smooth connection determined by the reference metric) are continuous. At interface boundaries, the continuity of these gradients (which are themselves tensors) is defined by the Jacobians $J^{A\alpha i}_{B\beta j}$ in the same way it is defined for any tensor field.

A reference metric is therefore an extremely useful (if not essential) tool for defining and enforcing continuity of tensor fields and their derivatives on multicube representations of manifolds. Unfortunately there is (at present) no straightforward way to construct these reference metrics on manifolds with arbitrary topologies. The examples given to date in the literature have been limited to manifolds with simple topologies where explicit formulas for smooth metrics were already known [19]. The purpose of this paper is to present a general approach for constructing suitable reference metrics for arbitrary manifolds. The goal is to develop a method that can be implemented automatically by a code using as input only the multicube structure of the manifold, i.e., from a knowledge of the collection of regions $\mathcal{B}_A$ and the way these regions are connected together by the interface maps $\Psi^{A\alpha}_{B\beta}$.

In this paper we develop, implement, and test a method for constructing positive-definite (i.e., Riemannian) C^1 reference metrics for compact, orientable two-dimensional manifolds with arbitrary topologies. While C^∞ reference metrics might theoretically be preferable, C^1 metrics are all that are required to define the continuity of tensor fields and their derivatives. We show in Appendix A that any C^1 reference metric provides the same definitions of continuity of tensor fields and their derivatives across interface boundaries as a C^∞ reference metric. This level of smoothness is all that is needed to provide the appropriate interface boundary conditions for the solutions of the systems of second-order PDEs most commonly used in mathematical physics. For all practicable purposes, therefore, C^1 reference metrics are all that are generally required.

Our method of constructing a reference metric $\tilde{g}_{ij}$ on Σ is built on a collection of star-shaped domains $\mathcal{S}_I$ with $I = 1, 2, \ldots, N_S$ that surround the vertex points $\mathcal{V}_I$, which make up the corners of the multicube regions. The star-shaped domain $\mathcal{S}_I$ is composed of copies of all the regions $\mathcal{B}_A$ that intersect at the vertex point $\mathcal{V}_I$. The interface boundaries of the