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Efficient sensitivity analysis method for chaotic dynamical systems

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ABSTRACT

The direct differentiation and improved least squares shadowing methods are both developed for accurately and efficiently calculating the sensitivity coefficients of time averaged quantities for chaotic dynamical systems. The key idea is to recast the time averaged integration term in the form of differential equation before applying the sensitivity analysis method. An additional constraint-based equation which forms the augmented equations of motion is proposed to calculate the time averaged integration variable and the sensitivity coefficients are obtained as a result of solving the augmented differential equations. The application of the least squares shadowing formulation to the augmented equations results in an explicit expression for the sensitivity coefficient which is dependent on the final state of the Lagrange multipliers. The LU factorization technique to calculate the Lagrange multipliers leads to a better performance for the convergence problem and the computational expense. Numerical experiments on a set of problems selected from the literature are presented to illustrate the developed methods. The numerical results demonstrate the correctness and effectiveness of the present approaches and some short impulsive sensitivity coefficients are observed by using the direct differentiation sensitivity analysis method.

1. Introduction

Sensitivity analysis is used to evaluate the rate of change of performance measure with respect to system parameter changes. The knowledge of sensitivity for dynamical systems is of considerable interest in structural dynamic reduction [1], optimal control [2], reliability analysis [3], and uncertainty analysis [4]. The sensitivity analysis can be carried out by using different methods such as the finite difference method, the direct differentiation method, and the adjoint variable method [5, 6]. The sensitivity of performance measure computed using the direct and adjoint methods is essentially identical. Although the finite difference method is easy to implement but it suffers from computational inefficiency and possible errors. On the contrary, the adjoint variables can be successively or repeatedly used for the evaluation of the sensitivity for different optimization variables. More information on the subject and the sensitivity theory can be found in the several classical books [7–9].

Many studies have been devoted to investigate the sensitivities of periodic solutions for nonlinear systems [10]. For instance, Liao [11] studied the sensitivity and robust stability of periodic motions for fractional order nonlinear dynamic systems by means of constrained optimization harmonic balance method [12]. Since time delay has an important effect

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on the performance of dynamic systems, research activities on periodic motions and dynamics of time delayed system have been carried out in [13] and the sensitivities of the Duffing oscillator subjected to various kinds of time delays are analyzed. In [14], the sensitivity analysis is extended to fractional-order nonlinear system with delay as nonlinear fractional delay differential equations are a very recent topic. Recently, the sensitivities of an airfoil system with various nonlinear structures are studied in [15].

Studying dynamical behavior of chaotic systems has attracted considerable attention in recent years. It is well known that sensitive dependence characterizes the unpredictability of chaotic phenomenon and trajectories adjacent to each other in chaotic systems become divergent in finite time [16]. Due to the extreme initial condition sensitivity, conventional approaches of sensitivity analysis are inadequate for solving the sensitivity problem of chaotic dynamical systems and this category sensitivity problem becomes an open question [17,18].

Mathematically, the sensitivity analysis problem of chaotic dynamical system is known to be ill-posed [19]. Ill-posed problems are inherently unstable and very sensitive to the inaccuracy in input data. Therefore, development of new computational schemes is necessary to overcome the numerical ill-posed problem.

In scientific and engineering community, considerable attention has been paid to the study of the long time statistical averaged quantities. Until now, relatively little work has been done addressing sensitivity computation of statistical quantities in chaotic dynamical systems since the distance between neighboring trajectories grows exponentially with time. For example, Wang [20] originally presented the Lyapunov eigenvector decomposition method to obtain the dynamic response sensitivity of chaotic dynamical systems. Although the Lyapunov eigenvector decomposition method has been found to be successful to predict the sensitivity coefficients of chaotic dynamical systems, but it has some potential limitations. The cost associated with this method is proportional to the number of Lyapunov exponents multiplied by the system dimension. The increase in the system dimension can result in a significant increase in the computational expense.

To alleviate the computational cost, a method named the Least Squares Shadowing (LSS) method has been developed in [21] to predict the sensitivity coefficients of chaotic limit cycle systems. Motivated by the pseudo-orbit shadowing theory in dynamical systems, the sensitivity analysis is transformed into a minimization problem for a least-squares objective function.

As an essential part of stability and ergodic theory, the shadowing property (also called the pseudo orbit tracing property) means that pseudo orbits generated by integration of the system equations are shaded by true orbits of the original system and therefore the numerically detected behavior of the system indeed reflects its real dynamical behavior. The main aim of the shadowing property is to obtain shadowing of approximate trajectories in a given dynamical system by true orbits of the system while the shadowing lemma establishes the existence of a true trajectory that remains close to a given pseudo trajectory. In addition, it is worth noting here that the ergodic hypothesis in chaotic systems should be satisfied when the shadowing theory is applied. For more clear and complete knowledge about the shadowing theory can be found in the monographs [22,23].

Based on the shadowing Lemma, traditional mathematical shadowing theory is devised for hyperbolic systems, which is characterized by the presence of expanding and contracting directions for derivatives. However, most physical systems (e.g. Earth's atmosphere) are non-hyperbolic. Although there is evidence that non-hyperbolic systems also have the shadowing property, however, it is very difficult to prove that the shadowing theory can be applied to the non-hyperbolic systems. Thus, finding the shadowing trajectories for non-hyperbolic systems has been the desire for many researchers.

Following the work of Wang [21], the convergence of the LSS method has been studied by Wang [24]. In a subsequent paper, convergence conditions are presented for the LSS method in [25]. However, the need for solving a large linear system of the LSS approach places a restriction on the hardware resources as large amounts of memory are required.

The multigrid-in-time technique for the LSS method has been investigated in [26] and several multigrid-in-time schemes including classic geometric multigrid, matrix restriction multigrid and solution restriction multigrid are considered. However, the computational cost and memory requirements due to a large number of time steps are still prohibitively large.

The checkpointing techniques which store only these solutions states corresponding to a small number of specific time steps known as check points are employed in [27]. However, there is no universal rule to select check points. Therefore, there is a need to develop the efficient algorithms to reduce the computational cost and memory requirements for the LSS scheme and subsequently improve the overall computational effort.

The motivation of the present paper is to develop the sensitivity computation methods for nonlinear dynamical systems. The developed approaches are aimed at calculation of sensitivity coefficients associated with time averaged quantity. The time averaged integral term is converted into differential equation and the augmented differential equations are then constructed to predict the sensitivity coefficients.

The remaining sections of the article are laid out as follows: Section 2 outlines the augmented equations of motion behind the sensitivity analysis problem and the direct differentiation formulation used to compute the sensitivity coefficients is provided in Section 3. In Section 4, the modified least squares shadowing method based on the augmentation equations of motion is derived and the algorithm used to implement the procedure computationally is given in Section 5. Numerical results along with some discussion and analysis are presented in Section 6 to verify the developed method. Finally, the paper ends with a summary highlighting the most significant contributions in Section 7.