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Transparent boundary conditions for iterative high-order parabolic equations

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ABSTRACT

Recently a new approach to the construction of high-order parabolic approximations for the Helmholtz equation was developed. These approximations have the form of the system of iterative parabolic equations, where the solution of the n-th equation is used as an input term for the $(n+1)$-th equation. In this study the transparent boundary conditions for such systems of coupled parabolic equations are derived. The existence and uniqueness of the solution of the initial boundary value problem for the system of iterative parabolic equations with the derived boundary conditions are proved. The well-posedness of this problem is also established and an unconditionally stable finite difference scheme for its solution is proposed.

1. Introduction

The wide-angle parabolic equations (WAPEs) presently are considered a main computational tool for many problems of wave propagation [1,2]. The most important applications probably include geophysics [3], radiowave propagation problems [1], and acoustics [2], to mention only a few. The WAPEs are traditionally derived by means of the operator square root approximation with a Padé series (hereafter they are referred to as Padé WAPE). Recently another approach to the wide-angle parabolic approximations was proposed in [4]. This derivation relies upon the systematic use of the multiple-scale expansion method, and the resulting high-order parabolic approximations have the form of the system of parabolic equations (PEs), where the input term of the n-th PE is obtained from the solution of $(n-1)$-th PE [4] (hence we call them iterative PEs, and this reflects the nature of the numerical algorithm required to solve such system). These new parabolic approximations have some advantages over the classical Padé-style approach to the WAPE derivation, which are discussed in great detail in [4]. The most important feature of iterative PE approach is the clear and consistent derivation of interface and boundary conditions for the case when the interfaces and boundaries have complicated shape [4]. By contrast, in classical Padé WAPE theory it is still quite unclear which conditions should be used for the boundary conditions at the curved or sloping interfaces, excepting for a number of very special cases [5,6]. This shortcoming restricts one to the use of the staircase approximation of a complicated interface or boundary in various applications. This issue is especially important in the acoustical applications, where the interfaces between the media are ubiquitous [2].

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In order to solve numerically the practical problems of wave propagation on the unbounded domains using these new parabolic approximations from [4] one has to truncate the domain with artificial boundaries. For more than two decades many research efforts were made to develop the methods of the artificial domain truncation for the Schrödinger-type equations (including the acoustical parabolic equation and optical paraxial equation). This domain truncation may be accomplished either by imposing the transparent boundary conditions (TBC) or by extending the computational domain with so-called perfectly matching layers (PMLs). The theory of the TBCs for the Schrödinger-like equations originated from the papers of Baskakov and Popov [7] and Papadakis [8]. For a review of different approaches to TBCs and PMLs see [9] and the numerous references therein. There are also some works concerning the TBCs for the conventional Taylor and Padé WAPEs, e.g. TBCs for the high-order Taylor WAPEs were proposed in [10], the TBCs for the rational–linear WAPEs were derived in [11,12] while the case of general Padé WAPE is dealt with in [13–15].

In this paper we derive the TBCs for the parabolic approximations proposed in [4]. These conditions are basically a natural but non-trivial generalization of the classical Baskakov–Popov TBCs [7]. We also show that the initial-boundary value problems (IBVPs) for the coupled PEs constituting the system from [4] with the derived TBCs are well posed and that its solution coincides with the solution of the same system on the unbounded domain. The derived TBCs may be also used for the solution of the wide angle mode parabolic equations.

We also propose a finite-difference scheme for the solution of the PEs system from [4] supplied with the derived TBCs. It is again a generalization of the numerical scheme of Baskakov and Popov [7]. In the interior of the computational domain the PEs are discretized using a second-order implicit Crank–Nicholson finite-difference method which is unconditionally stable for the unbounded domain or homogeneous Dirichlet conditions [9]. The incorporation of the TBCs into a numerical scheme may however render it only conditionally stable [16]. Sun and Wu [17] proved however that the Baskakov and Popov discretization of the TBCs leads to an unconditionally stable scheme. We adapted their proof for our new numerical scheme which also turns out to be unconditionally stable.

2. Wide angle parabolic approximations

Let us consider the problem of sound propagation in a 2D acoustical waveguide $\Omega = \{(x,z)|z \geq 0\}$ consisting of the water layer and one or more layers of bottom (which is assumed to be liquid), where z is the depth and x is the horizontal variable (here we use the acoustical notation following [4], although the same results may be reproduced for the open waveguides in optics and radiowave propagation theory). The acoustical pressure $p(x,z)$ due to a point source located at $x = 0$, $z = z_s$ then satisfies the Helmholtz equation

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial p}{\partial z}\right) + \frac{1}{\rho}\kappa^2 p = \frac{1}{\rho}\delta(x)\delta(z - z_s)\,, \tag{2.1}$$

where the medium parameters are the density $\rho = \rho(x,z)$ and the wavenumber $\kappa = \kappa(x,z) = \omega^2/c^2$. A pressure-release Dirichlet-type boundary condition

$$p(x,0) = 0 \tag{2.2}$$

is usually imposed at the ocean surface $z = 0$, while for sufficiently large values of depth (say for $z \geq L$) the medium is assumed to be homogeneous, i.e. $\rho(x,z) = \rho_b$ and $\kappa(x,z) = \kappa_b$ for all $z \geq L$. In order to correctly set a BVP for the equation (2.1) one also requires certain radiation conditions to be satisfied at infinity $R = \sqrt{x^2 + z^2} \to \infty$ (see e.g. [2]). The propagation problems in the shallow-water acoustics usually feature additional complication associated with the presence of interfaces, i.e. surfaces $z = H(x)$ where media parameters have finite jump discontinuities (e.g. water-bottom interface). The following coupling conditions are imposed at the interface $z = H(x)$ (see e.g. [2]):

$$p|_{z=H(x)+0} = p|_{z=H(x)-0}\,, \quad \left.\frac{1}{\rho}\frac{\partial p}{\partial \mathbf{n}}\right|_{z=H(x)+0} = \left.\frac{1}{\rho}\frac{\partial p}{\partial \mathbf{n}}\right|_{z=H(x)-0}\,. \tag{2.3}$$

The bottom relief described by the function $z = H(x)$ in practical problems is often very complicated.

Usually the BVP (2.1)–(2.2)–(2.3) is too complicated to be solved directly, and the high-order parabolic equations are used [2] to approximate the solution of the Helmholtz equation (2.1). While sacrificing some relatively unimportant propagation features, thus we obtain mathematical formulation of the problem which is much more efficient and easier for the numerical implementation.

Among the shortcomings of the traditional approach to the PEs derivation (the one based on the operator square root approximation) is its inability to systematically account for the sloping and variable bottoms. Strictly speaking the very Helmholtz operator factorization which leads to the equations for the forward- and backward-propagating waves (containing the operator square root) relies upon the assumption of waveguide range-independence [2]. It is also well-known that even the simplest boundary and interface conditions for the case of the sloping bottom may lead to the ill-posedness of the IBVP for the parabolic equation [5]. Although some efforts we made to derive the proper interface and boundary conditions for the PEs in the case of the sloping bottoms [6,5], to our knowledge they were only partially successful. For example, in the paper [6] authors consider the case of the rational–linear wide-angle PE and the sloping pressure-release bottom, while for the general n-th order Padé WAPE and the arbitrarily sloping penetrable bottom no interface conditions were proposed