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Asymptotic analysis of discrete schemes for non-equilibrium radiation diffusion

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ABSTRACT

Motivated by providing well-behaved fully discrete schemes in practice, this paper extends the asymptotic analysis on time integration methods for non-equilibrium radiation diffusion in [2] to space discretizations. Therein studies were carried out on a two-temperature model with Larsen's flux-limited diffusion operator, both the implicitly balanced (IB) and linearly implicit (LI) methods were shown asymptotic-preserving. In this paper, we focus on asymptotic analysis for space discrete schemes in dimensions one and two. First, in construction of the schemes, in contrast to traditional first-order approximations, asymmetric second-order accurate spatial approximations are devised for flux-limiters on boundary, and discrete schemes with second-order accuracy on global spatial domain are acquired consequently. Then by employing formal asymptotic analysis, the first-order asymptotic-preserving property for these schemes and furthermore for the fully discrete schemes is shown. Finally, with the help of manufactured solutions, numerical tests are performed, which demonstrate quantitatively the fully discrete schemes with IB time evolution indeed have the accuracy and asymptotic convergence as theory predicts, hence are well qualified for both non-equilibrium and equilibrium radiation diffusion.

1. Introduction

Non-equilibrium radiation diffusion problems, e.g., two-temperature models describing the non-equilibrium of gray radiation and material, play an important role in inertial confined fusion, Z-pinch and astrophysics researches. In practice, it may occur that thermal exchange is much faster than diffusion process in some regions. In such circumstance, the system is described by a one-temperature model, which is an asymptotic limit of two-temperature model since it unfolds an equilibrium limit case when the ratio of the absorption mean free path to the gradient length scale tends to zero. In numerical simulations, one expects to maintain this asymptotic property. Hence it is crucial to design discrete schemes appropriate for both non-equilibrium and equilibrium radiation diffusion problems.

In asymptotic analysis, a discrete scheme of the original problem is called to be asymptotic-preserving (AP) if it converges to a scheme consistent with the limit problem when scaling parameter (conventionally denoted by ε, etc.) tends to zero [1], which makes it available for the original and limit problem with flexible discretization parameters, hence applicable to both asymptotic and non-asymptotic regions.

There are many publications describing asymptotic analysis for transport and hydrodynamic problems [1–8]. For example, a summary on AP schemes for plasma fluid models was presented in [3]. Some AP finite volume schemes for hyperbolic

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heat equations were established on unstructured meshes in [9]. A class of high order AP DG schemes was developed for discrete-velocity kinetic equations in [10]. However there are seldom studies on asymptotic analysis for radiation diffusion problems, despite of their essential roles in radiation multi-physics. In [11], the classic Marshak wave equation and a higher order equilibrium diffusion approximation were studied for a time-dependent non-equilibrium radiative transfer system.

For non-equilibrium radiation diffusion, asymptotic analysis for IB and LI time integrations was presented in [2]. Their asymptotic-preserving property was demonstrated by theoretical analysis and qualitative numerical tests. More studies on numerical schemes considering time accuracy and iteration algorithms can be found in [12–15]. Also there are some papers demonstrating the second order spatial accuracy, i.e., $O(h^2)$ spatial accuracy where h is a nominal grid size and as $h \to 0$ (we assume this is implied for the rest of the paper), of finite volume schemes. But for non-equilibrium radiation diffusion with Larsen's flux-limiter, little attention has been paid on the sensitivity of the global accuracy to the boundary flux-limiter treatment. If a first order, i.e., $O(h)$, spatial approximation for flux-limiter is simply applied on boundary, some numerical examples can reach second order accuracy, while others can not. Moreover, to our knowledge, there is no asymptotic analysis for spatially discrete schemes or fully discrete schemes for such problem. Although being regarded as correct intuitively, the asymptotic property of the former has never been proved rigorously.

To get well-behaved fully discrete schemes for non-equilibrium radiation diffusion, based on the work in [2], this paper is devoted to developing spatial schemes with guaranteed second order accuracy and first order asymptotic-preserving property (a description of the AP order though $O(\varepsilon^n)$ as $\varepsilon \to 0$ is reviewed later in Section 5). By combining the studies on the temporally and spatially discrete schemes, the conclusion for the fully discrete schemes comes into existence as a corollary.

The work is carried out on two-temperature models with Larsen's flux-limited diffusion operator in dimensions one and two. Asymptotic-preserving discrete schemes with second order spatial accuracy are proposed, which fit both one-temperature and two-temperature models. In construction of the schemes, to assure the second order consistency errors on the entire computation domain, we apply asymmetric second-order spatial discrete approaches for flux-limiters on boundary and symmetric schemes at interior points. This simple treatment effectively improves the accuracy order, and significantly decreases the error data. The first order asymptotic-preserving property of the schemes is demonstrated theoretically via formal analysis.

There are some works on artificial solutions for linear diffusion and transport problems [16–19] where the classical strategy using Laplace or Fourier transformation for temporal or spatial variant was applicable. However, it is rather difficult to construct exact solution models for scaled non-equilibrium radiation diffusion problems due to the strong nonlinearity and the additional scaling parameter. In this paper, we devise some exact solution problems with characteristics of scaling parameter and strong nonlinearity. Using the delicate numerical examples with these manufactured solutions and reference solutions, we carry out numerical experiments to verify the performance of the schemes qualitatively and quantitatively. The results show that the fully discrete schemes obtained by combining the spatial discretizations with IB temporal evolution have second order accuracy and expected behavior in asymptotic limit. The convergence property is not affected by time and space step lengths when the scaling parameter tends to zero, hence these schemes adapt to both non-equilibrium and equilibrium radiation diffusion.

The paper is organized as follows. In Section 2, we review one-dimensional non-equilibrium radiation diffusion problem and give three spatially discrete schemes for its scaled problem. In Section 3, we present an asymptotic analysis for these schemes, and show they have first order asymptotic-preserving property. The asymptotic property is also valid for the fully discrete schemes combining them with IB and LI temporal discretizations. Then in Section 4, the asymptotic analysis is extended to two-dimensional problem. Next, numerical tests are carried out in Section 5 to validate the theoretical results. Finally conclusions are given in Section 6.

2. Spatially discrete schemes for non-equilibrium radiation diffusion

Consider the one-dimensional non-equilibrium radiation diffusion [2]

$$
\begin{aligned}
&\frac{\partial E}{\partial t} - \frac{\partial}{\partial x}\left(D_R \frac{\partial E}{\partial x}\right) = c\sigma_a (B - E),\\
&C_v \frac{\partial T}{\partial t} = -c\sigma_a (B - E),\\
&\sigma_a = \bar{\sigma}_a T^{-3},\\
&B = a_R T^4,
\end{aligned}
\tag{2.1}
$$

where E is the radiation energy density, T is the material temperature, c is the speed of light, σ_a is the absorption opacity, C_v is the material heat capacity, $\bar{\sigma}_a$ is a parametric constant, and a_R is the radiation constant. In the original model, the diffusion coefficient takes the form $D_R = \frac{c}{3\sigma_a}$. It may occur that the energy flow spreads faster than light. To avoid