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A discretization of the multigroup P_N radiative transfer equation on general meshes

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ABSTRACT

We propose and study a finite volume method of discrete duality type for discretizing the multigroup P_N approximation of radiative transfer equation on general meshes. This method is second order-accurate on a very large variety of meshes, stable under a Courant–Friedrichs–Lewy condition and it preserves naturally the diffusion asymptotic limit.

1. Introduction

Numerous physics phenomena (among other particle transport) are modelled by the *Boltzmann* equation. Because this seven-dimensional equation (three space variables, three velocity variables and time) is rather difficult to solve, several types of approximation have been devised and numerically tested since about fifty years. The most commonly used are the diffusion, flux-limited diffusion or variable Eddington factors diffusion approximations, the discrete ordinates (S_N), the spherical harmonics (P_N) or simplified spherical harmonics (SP_N) and the implicit Monte-Carlo methods (see [1] and the thorough developments of the reference books [2,3]). Moreover, in order to take into account the mutual particle material interactions, the Boltzmann equation has to be coupled with the hydrodynamics equations that model the background material behaviour. Consequently, when *Lagrangian* hydrodynamics equations are involved, the Boltzmann equation has to be approximated on (eventually) distorted Lagrangian meshes that match the material motion.

In the present paper we have chosen the P_N approximation. So we are concerned with the numerical approximation of the P_N Boltzmann equation on *general* (that is unstructured, polygonal, distorted, non-conformal or even non-convex) meshes.

For about twenty years there have been great efforts to devise *compatible* discretizations of partial differential operators that work on general meshes, particularly for elliptic problems but also for Maxwell or Navier–Stokes equations (see for example [4,5] and the references therein). Among other compatible methods one can quote the *discrete duality finite volume* (DDFV, [6–9]) and the *mimetic finite volume* [10] methods. Recall that the word "compatible" means that the discrete operators have to satisfy discrete counterparts of characteristic properties of the continuous operators, as the Green's formula, the symmetries, the preservation of the energy or the involutions for example. For ten years we proposed a compatible DDFV type method for solving the two-dimensional Maxwell equations and, more generally, two-dimensional wave type equations, on nearly arbitrary meshes for which a stability analysis has been carried out: see [11–13]. This method has

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been also used for dealing with the rotating shallow water equations for which a dispersion analysis has been carried out: see [14]. Our purpose is to show that this type of finite volume method can be fruitfully widened to the multigroup P_N approximation of the Boltzmann equation. Indeed we obtain a new finite volume method which enjoys several attractive features:

1. use of general meshes,
2. second-order experimental accuracy on a broad variety of meshes,
3. stability under a Courant Friedrichs Lewy (CFL) condition,
4. preservation of the diffusion asymptotic limit,
5. generalization of either legacy methods as the MAC and Yee's schemes [15,16] for the approximation of wave-like equations on rectangle meshes or the recent so-called *Staggered grid Radiation Moment Approximation* (STARMAP) method for the approximation of P_N equations on rectangle meshes [17,18].

To our knowledge devising methods combining all these features is a rather challenging task: see the recent works [19, 20] where a Godunov-type method is proposed for solving the hyperbolic heat equation and, more generally, the abstract *Friedrichs* systems of equations on unstructured meshes.

We will focus here on the Boltzmann equation for photons, namely the *radiative transfer* equation, but of course other type of particles could be dealt with as well. To facilitate the exposition we will consider only reflecting boundary conditions. Furthermore we will suppose that the coupling between the Boltzmann and hydrodynamics equations is done by a splitting method so that the material density and temperature will be supposed given.

The organization of the paper is as follows. Section 2 is devoted both to fix the notation and to recall some prominent features of the *multigroup* P_N approximation of the radiative transfer equation. We remind briefly the reader of the definition of Legendre and spherical harmonics functions and their recursion properties in Appendix A. The reflecting boundary conditions in the P_N framework are established in Appendix D. Furthermore we prove that the P_N equations admit an asymptotic diffusion limit (Theorem 1). The finite volume method we propose is described in section 3 in the two-dimensional framework. Section 3.1 is devoted to the DDFV discretization of differential operators on general meshes and to some of its compatibility properties (namely a discrete Green's formula and discrete differential identities, see Appendix E). Sections 3.1 and 3.2 detail respectively the space and time discretizations which are summarized in Appendix F under a pseudo-code formulation. In section 3.4 the L^2 stability of the scheme is studied (Theorems 2 to 4) while we prove that it is asymptotic preserving in section 3.5 (Theorem 5). In section 3.6 it is proved that the scheme coincides with the STARMAP method [18] when rectangular meshes are used (Theorem 6). Finally three numerical experiments are presented in section 4, the first one allowing us to assess the experimental order of convergence with a manufactured solution and the other being reference benchmarks borrowed from the literature [1,21,18]. Concluding remarks follow in section 5.

In all what follows the vectors and the matrices will be denoted by bold letters while $\boldsymbol{x} = (x, y, z)$, S^2, t, δ_m^n, $\boldsymbol{I}_n$ and i will stand respectively for the position vector, the unit sphere in R^3, the time, the delta Kronecker function, the unit $n \times n$ matrix and the imaginary unit number.

2. The radiative transfer equation and its multigroup P_N approximation

Let c be the speed of light and ν, $\boldsymbol{\omega}$ the frequency and direction of the photons. We use the standard spherical parametrization:

$$\boldsymbol{\omega} = (\cos\psi\sqrt{1-\mu^2}, \sin\psi\sqrt{1-\mu^2}, \mu),$$

where ψ is the *azimuthal* angle ($0 \le \psi < 2\pi$) and $\mu = \cos\beta$, β being the *polar* angle ($0 \le \beta \le \pi$), see Fig. 1.

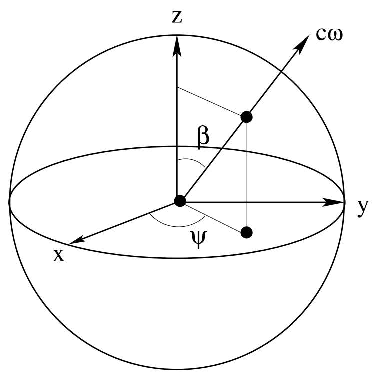

Fig. 1. Spherical coordinates for the velocity.