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Short note

On velocity errors due to irrotational forces in the Navier–Stokes momentum balance

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ABSTRACT

This contribution studies the influence of the pressure on the velocity error in finite element discretisations of the Navier–Stokes equations. Four simple benchmark problems that are all close to real-world applications convey that the pressure can be comparably large and is not to be underestimated. In fact, the velocity error can be arbitrarily large in such situations. Only pressure-robust mixed finite element methods, whose velocity error is pressure-independent, can avoid this influence. Indeed, the presented examples show that the pressure-dependent component in velocity error estimates for classical mixed finite element methods is sharp. In consequence, classical mixed finite element methods are not able to simulate some classes of real-world flows, even in cases where dominant convection and turbulence do not play a role.

1. Introduction

Often, spurious velocity oscillations in discretisations of the incompressible Navier–Stokes equations, where one looks for a velocity field $\mathbf{u}$ and a pressure p in space dimension $d \in \{2, 3\}$, are attributed to dominant convection at high Reynolds numbers, like in the case of *scalar* singularly perturbed advection–diffusion equations. However, spurious velocity oscillations can also be excited by a different mechanism, which can only appear in *vector equations*: the lack of L^2-orthogonality of (only) discretely divergence-free vector fields and large gradient fields in the Navier–Stokes momentum balance [6], which is sometimes observed and addressed in the literature as poor mass balance [8,9]. This lack of L^2-orthogonality makes the velocity error between the exact velocity $\mathbf{u}$ and the discrete velocity $\mathbf{u}_h$ of mixed finite elements like the Taylor–Hood element for the incompressible Stokes equations *pressure-dependent*, i.e. $C_2 \geq 1$ in

$$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L^2} \leq C_1 \inf_{\mathbf{w}_h \in V(\mathcal{T})} \|\nabla(\mathbf{u}-\mathbf{w}_h)\|_{L^2} + \frac{C_2}{\nu} \inf_{q_h \in Q(\mathcal{T})} \|p - q_h\|_{L^2}, \tag{1}$$

which can be found in textbooks like [3,1]. Here, $V(\mathcal{T})$ and $Q(\mathcal{T})$ denote the discrete velocity and discrete pressure trial spaces, $\|\cdot\|_{L^2}$ denotes the scalar/vectorial L^2 norm, and ν denotes a possibly (very) small parameter (for example the inverse of the Reynolds number). Please note that similar velocity error estimates can also be found for more complicated flow problems like the steady Navier–Stokes case with small data assumption [4]. Also in these cases, the velocity-dependent error part will vanish, whenever the exact velocity lies in the discrete velocity space, and the pressure-dependent error part will be qualitatively the same.

Although mixed finite elements like the Scott–Vogelius element exist, whose velocity error is *pressure-independent* ($C_2 = 0$), traditionally the pressure-dependence of the velocity error of many flow discretisations seems to be regarded of

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secondary importance. More or less, common belief in the numerical analysis community is that convergence of asymptotically optimal order would be sufficient for potential success in real-world flow problems. However, we want to emphasize in this contribution that there is an assumption hidden in this belief: the pressure has to be comparably small and may not be too complicated, since otherwise the pressure-dependent part of the velocity error would be dominant and the constant C_2/ν in error estimate (1) could be arbitrarily large, depending on the flow problem. In order to practically demonstrate that the assumption of a small and simple pressure is generally wrong in real-world applications, we present four simple benchmarks, where the pressure is *large w.r.t. the velocity* and the velocity will be an (at most) quadratic vector field. In the first benchmark, a buoyancy force exactly balances the pressure gradient (momentum balance: $\nabla p = \mathbf{f}$, where $\mathbf{f}$ denotes a conservative exterior force), yielding a hydrostatic situation. Assuming that $\mathbf{f} \in P_k^d$ is a vector of polynomials of order k (abbreviated as P_k), we have $\mathbf{u} = \mathbf{0}$ and $p \in P_{k+1}$ in such a situation. In this sense, the pressure p is more complicated than the velocity $\mathbf{u}$. Obviously, such a conclusion can be extended to flows with a momentum balance $\nabla p \approx \mathbf{f}$, which one could call (quasi-)hydrostatic, as it appears in Boussinesq flows. In the second benchmark, a strong y-dependent Coriolis force balances the pressure gradient (momentum balance: $2\mathbf{\Omega} \times \mathbf{u} + \nabla p = \mathbf{0}$), as it appears in large-scale flows in the so-called β-plane approximation in meteorology [10] where $\mathbf{\Omega}$ is a linear polynomial. Then, the pressure is a polynomial of order $k+2$, if the velocity is a polynomial of order k. In the third and fourth benchmark, the nonlinear convection term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is balanced by the pressure gradient (momentum balance: $(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{0}$). In this case $\mathbf{u} \in P_k^d$ leads to $p \in P_{2k}$. These examples demonstrate that the error contribution $\frac{C_2}{\nu} \inf_{q_h \in Q(\mathcal{T})} \|p - q_h\|_{L^2}$ appears in all kinds of incompressible flows and can have a major influence on the velocity error. In fact, in all four examples, this error contribution is the *only error source*, since the continuous velocity solution lies in the discrete velocity space.

Since the pressure is in all four examples comparably large, low-order mixed finite element methods on unstructured grids like the Taylor–Hood finite element method will heavily suffer from spurious velocity oscillations, as can be observed in the numerical experiments related to Figs. 2, 4, 6 and 8. Instead, pressure-robust ($C_2 = 0$ in (1)) finite element methods, like the Scott–Vogelius finite element method [11,12], will be able to deliver the exact velocity solution.

The last two examples, where the nonlinear convection term balances the pressure gradient, are especially important. In these examples, spurious velocity oscillations are excited, when the Reynolds number becomes large, though they are not excited by dominant advection. We remind the reader that even unstable Galerkin discretisations for singularly perturbed advection–diffusion equations deliver the exact solution, whenever the solution lies in the discrete trial space. Moreover, spurious oscillations due to dominant convection are only excited in the presence of (interior or boundary) layers, which our example does not have. Therefore, the last two benchmarks deliver simple and highly didactic examples demonstrating that the nonlinear convection term excites two different kinds of spurious velocity oscillations at high Reynolds numbers. This argument was made recently in [5], but it does not seem to be widely acknowledged by the CFD community, though its importance for the discretisation of the nonlinear convection term is potentially high.

2. The Navier–Stokes model problem and its discretisation

This section recalls the Navier–Stokes equations with Coriolis force and its discretisation with finite element methods.

2.1. The Navier–Stokes equations

The Navier–Stokes equations with angular velocity Ω, right-hand side $\mathbf{f}$ and Dirichlet data $\mathbf{u}_D$ for some $d \in \{2, 3\}$ dimensional bounded Lipschitz domain D with polyhedral boundary ∂D read

$$-\nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p + 2\Omega \times \mathbf{u} = \mathbf{f} \quad \text{in } D, \qquad \nabla \cdot \mathbf{u} = 0 \quad \text{in } D, \qquad \mathbf{u} = \mathbf{u}_D \quad \text{along } \partial D, \tag{2}$$

where $\mathbf{u_D}$ describes some smooth boundary data in our benchmark problems below. Note, that to understand the Coriolis force $2\Omega \times \mathbf{u}$ in the Navier–Stokes equations in 2D, the velocity $\mathbf{u}(x, y) = (u_1(x, y), u_2(x, y))^T$ is embedded into 3D by setting $\mathbf{u} = (u_1, u_2, 0)$. In this case angular velocities $\Omega = (0, 0, \omega(x, y))$ are regarded with some scalar function ω and result in $2\Omega \times \mathbf{u} = 2\omega(-u_1, u_2, 0)^T$ where the zero z-component is eliminated in the numerical computations, of course.

The weak formulation employs the multilinear forms

$$a(\mathbf{u}, \mathbf{v}) := \int_{\mathcal{T}} \nu\nabla\mathbf{u} : \nabla\mathbf{v}\, dx, \qquad b(\mathbf{u}, q) := -\int_D q\nabla \cdot \mathbf{u}\, dx,$$

$$c(\mathbf{u}, \mathbf{v}) := \int_D (2\Omega \times \mathbf{u}) \cdot \mathbf{v}\, dx, \qquad n(\mathbf{a}, \mathbf{u}, \mathbf{v}) := \int_D ((\mathbf{a} \cdot \nabla)\mathbf{u}) \cdot \mathbf{v}\, dx,$$

$$F(\mathbf{v}) := \int_D \mathbf{f} \cdot \mathbf{v}\, dx$$

and characterises weak solutions $(\mathbf{u}, p) \in H^1(D; \mathbb{R}^d) \times L_0^2(D)$ of (2) by $\mathbf{u} = \mathbf{u}_D$ along ∂D and