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Galerkin orthogonal polynomials

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ABSTRACT

The Galerkin method offers a powerful tool in the solution of differential equations and function approximation on the real interval $[-1, 1]$. By expanding the unknown function in appropriately chosen global basis functions, each of which explicitly satisfies the given boundary conditions, in general this scheme converges exponentially fast and almost always supplies the most terse representation of a smooth solution. To date, typical schemes have been defined in terms of a linear combination of two Jacobi polynomials. However, the resulting functions do not inherit the expedient properties of the Jacobi polynomials themselves and the basis set will not only be non-orthogonal but may, in fact, be poorly conditioned. Using a Gram-Schmidt procedure, it is possible to construct, in an incremental fashion, polynomial basis sets that not only satisfy any linear homogeneous boundary conditions but are also orthogonal with respect to the general weighting function $(1-x)^{\alpha}(1+x)^{\beta}$. However, as it stands, this method is not only cumbersome but does not provide the structure for general index n of the functions and obscures their dependence on the parameters (α, β). In this paper, it is shown that each of these Galerkin basis functions, as calculated by the Gram-Schmidt procedure, may be written as a linear combination of a small number of Jacobi polynomials with coefficients that can be determined. Moreover, this terse analytic representation reveals that, for large index, the basis functions behave asymptotically like the single Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$. This new result shows that such Galerkin bases not only retain exponential convergence but expedient function-fitting properties too, in much the same way as the Jacobi polynomials themselves. This powerful methodology of constructing Galerkin basis sets is illustrated by many examples, and it is shown how the results extend to polar geometries. In exploring more generalised definitions of orthogonality involving derivatives, we discuss how a large class of differential operators may be discretised by Galerkin schemes and represented in a sparse fashion by the inverse of band-limited matrices.

1. Introduction

Spectral methods are a widely used tool in the solution of differential equations, function approximation and variational problems [1,2]. Their utility is based on the fact that if the solution sought is smooth, usually only a few terms in an expansion of global basis functions are needed to represent it to high accuracy. This efficiency comes about because the spectral coefficients, f_n, typically tend to zero faster than any algebraic power of their index n, showing either exponential or sometimes super-exponential convergence [3]. On the non periodic canonical interval $[-1, 1]$, the Jacobi polynomials are a well-known class of polynomials exhibiting spectral convergence, of which particular examples are Chebyshev polynomials of the first and second kinds, and Legendre polynomials [3]. Chebyshev polynomials are often a popular choice since, via their links with Fourier methods, they have a fast transform.

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When approximating a given function by a spectral expansion, the choice of which Jacobi polynomials to use can rest on the required asymptotic behaviour of the error. For example, Chebyshev polynomials of the first kind are well-known to minimise the maximum error between any function and its approximant, the so-called minimax or L^∞ norm characteristic [4,5]. Such an expedient property has led to widespread use of Chebyshev approximants in numerical computation [6]. Lesser known but, in the appropriate case no less useful, is the property that the Chebyshev polynomials of the second kind minimise the error in the L^1 norm, that is, the integrated absolute error between the function and its approximant. These properties are intimately tied to certain properties of the polynomials themselves: Chebyshev polynomials of the first kind are equal-ripple (uniform oscillations) and those of the second kind are equal-area (the area under the curve between any two consecutive zeros is constant). Lastly, Legendre polynomials minimise the error between any function and its approximant in the L^2 norm although this is not associated with any obvious graphical property of the polynomials themselves.

Physical problems almost always involve known boundary conditions which can be fully exploited in a Galerkin method [7,8,3]. Such a scheme adopts an expansion in terms of a global basis set constructed so that each member explicitly satisfies the boundary conditions; by encoding this additional information, out of all numerical methods, this approach almost always provides the most terse numerical representation. If an analytic solution of a differential equation is known but difficult to compute, it is expedient to write it in terms of a spectral expansion (for instance in Chebyshev polynomials) which, after finding the coefficients, is easy to evaluate. In this paper, we shall see such an approximation method can be extended by using an expansion in terms of an exponentially convergent orthogonal Galerkin basis. Furthermore, as in functional approximation by Jacobi polynomials, the principal error stems from the first ignored term in the expansion which can be chosen in an optimal fashion, for instance, to be quasi-equal-ripple and therefore for the approximant to minimise the L^∞ error.

Unlike a spectral expansion of a known function, the error in the solution of a differential equation or variational problem is not well approximated by the first ignored term (since the error contaminates all coefficients). It is therefore not possible to prescribe in advance, by choice of the basis set, the asymptotic behaviour of the error. However, as we shall see subsequently, Galerkin schemes remain a useful tool since, not only do they converge exponentially fast but, because the boundary conditions are already encoded, in general they converge faster than canonical spectral methods. In a traditional Galerkin method, a differential equation is discretised by imposing an orthogonality condition to the same set of basis functions. However, other variants include imposing orthogonality to a different set of functions in the so-called Petrov-Galerkin scheme and, by extending the basis sets to those of compact support, Galerkin schemes form the foundation of the finite-element method [9]. By adopting a Galerkin expansion at the outset, often subsequent analysis is eased since the boundary conditions may, essentially, be dispensed with. By contrast, in other pseudospectral schemes that could be employed to solve differential equations (e.g. Chebyshev-tau or a collocation method), the boundary conditions are carried through to the end of the calculation where they are imposed explicitly as additional rows of the discretised matrix system.

There are several particular cases where Galerkin expansions have the greatest utility. First are problems where terseness of the solution is pivotal. Such a case can arise when forming low-order models of a system, or when using symbolic computation to produce an approximation to the solution. To expedite the solution of matrix problems symbolically, the matrix size should be reduced as much as possible, a property which Galerkin methods can readily provide. Second are variational problems, where often integration by parts of the raw equations produces awkward boundary terms. Unless one is very lucky, the boundary conditions cannot be used to evaluate these terms and no further progress is possible. Within a Galerkin method, such boundary terms can always be evaluated and a matrix system then constructed [10]. Third, Galerkin methods often exhibit the lowest condition number dependence on matrix size. Such an issue may arise when solving a problem to very high resolution. For instance, although a standard Chebyshev-tau method may theoretically be capable of resolving a fine-scale solution, its numerical discretisation may be too ill-conditioned and any answer swamped with numerical error in finite precision (although, solving the system using high precision will give an accurate answer). Galerkin methods often have a low scaling of the condition number with matrix size, thus minimising the computational error and allowing high resolution in finite precision. The main drawback of Galerkin methods is that, in general, no fast-transform exists and, until now, there has been no generally accepted method of constructing the required basis sets for arbitrary boundary conditions.

Galerkin schemes are easily constructed when considering linear homogeneous boundary conditions. Note that if the given boundary conditions are not homogeneous they can always be made so with the addition of an appropriate function to the unknown solution, with the associated modification of the equations. To date, typical schemes involve forming a linear combination of a Jacobi polynomial (usually a Chebyshev polynomial, $T_n(x)$) with one of neighbouring index or some fixed low-order polynomial in order to satisfy the required conditions [11,7,3]. For example, the following are two possible choices of basis sets that satisfy the boundary condition $f'(1) = 0$:

$$\phi_n(x) = T_n(x) - n^2 T_1(x), \qquad \chi_n(x) = (n-1)^2 T_n(x) - n^2 T_{n-1}(x).$$

It is clear that $\phi_n(x)$ becomes increasingly ill-conditioned as n increases since, when normalised, $\phi_n(x) \to T_1(x) = x$ as $n \to \infty$ which is independent of n. The second case, $\chi_n(x)$, is better conditioned but forms a basis set that is neither orthogonal nor close to equal-ripple. Thus in recombining Chebyshev polynomials, many of their optimal properties have been lost.

An alternative method to construct a basis set is to use a Gram-Schmidt procedure in the following way. The lowest-degree polynomial that satisfies the boundary conditions is $\Psi_1(x) = 1$ (up to a normalisation). The next element $\Psi_2(x)$ is written as an arbitrary quadratic in x, whose coefficients are determined by imposing (i) the boundary condition and (ii) orthogonality to $\Psi_1(x) = 1$. Note that we need to jump degree from 0 to 2: there is no non-trivial linear form that will satisfy