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A new anisotropic mesh adaptation method based upon hierarchical a posteriori error estimates

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ABSTRACT

A new anisotropic mesh adaptation strategy for finite element solution of elliptic differential equations is presented. It generates anisotropic adaptive meshes as quasi-uniform ones in some metric space, with the metric tensor being computed based on hierarchical a posteriori error estimates. A global hierarchical error estimate is employed in this study to obtain reliable directional information of the solution. Instead of solving the global error problem exactly, which is costly in general, we solve it iteratively using the symmetric Gauß–Seidel method. Numerical results show that a few GS iterations are sufficient for obtaining a reasonably good approximation to the error for use in anisotropic mesh adaptation. The new method is compared with several strategies using local error estimators or recovered Hessians. Numerical results are presented for a selection of test examples and a mathematical model for heat conduction in a thermal battery with large orthotropic jumps in the material coefficients.

1. Introduction

Anisotropic mesh adaptation has proved to be a useful tool in numerical solution of partial differential equations (PDEs). This is especially true when problems arising from science and engineering have distinct anisotropic features. The ability to adapt the size, shape, and orientation of mesh elements according to certain quantities of interest can significantly improve the accuracy of the solution and enhance the computational efficiency.

Criteria for an optimal anisotropic triangular mesh were already given by D'Azevedo [1] and Simpson [2] in the early nineties of the last century. A number of algorithms for automatic construction of such meshes have since been developed.

A common approach for generating an anisotropic mesh is based on generation of a quasi-uniform mesh in some metric space. A key component of the approach is the determination of an appropriate metric often based on some type of error estimates. Unfortunately, classic isotropic error estimates do not suit this purpose well because they generally do not take the directional effect of the error or solution derivatives into consideration. This explains the recent interest in anisotropic error estimation; for example, see anisotropic interpolation error estimates by Formaggia and Perotto [3], Huang [4], and Huang and Sun [5]. Such error estimates for numerical solution of PDEs can be found, among others, in works by Apel [6], Kunert [7], Formaggia and Perotto [8], and Picasso [9].

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It is worth pointing out that most existing anisotropic error estimates are a priori, requiring information of the exact solution of either the underlying problem or its adjoint, which is typically unavailable in a numerical simulation. A widely-used approach of avoiding this difficulty in practical computation is to replace the information by one recovered from the obtained numerical approximation. A number of recovery techniques can be used for this purpose, such as the gradient recovery technique by Zienkiewicz and Zhu [10,11] and the technique based on the variational formulation by Dolejší [12]. Zhang and Naga [13] have recently proposed a new algorithm to reconstruct the gradient (which can also be used to reconstruct the Hessian) by fitting a quadratic polynomial to the nodal function values and subsequently differentiating it. It has been shown by Zhang and Naga [13] and by Vallet et al. [14] that the latter is robust and works best among several recovery techniques. Generally speaking, recovery methods work well when exact nodal function values are used but may lose some accuracy when applied to finite element approximations on non-uniform meshes. Nevertheless, the optimality of mesh adaptation based on those recovered approximations can still be proven under suitable conditions, see Vassilevski and Lipnikov [15]. More recently, conditions for asymptotically exact gradient and convergent Hessian recovery from a hierarchical basis error estimator have been given by Ovall [16]. His result is based on superconvergence results by Bank and Xu [17,18], which require that the mesh be uniform or almost uniform.

The objective of this paper is to study the use of a posteriori error estimates in anisotropic mesh adaptation. Although a posteriori error estimates are frequently used for mesh adaptation, especially for refinement strategies and recently also for construction of equidistributing meshes for numerical solution of two-point boundary value problems by He and Huang [19] as well as in connection with the moving finite element method by Lang et al. [20], up to now only few methods for their use in anisotropic mesh adaptation have been published. For example, Cao et al. [21] studied two a posteriori error estimation strategies for computing scalar monitor functions for use in adaptive mesh movement; Apel et al. [22] investigated a number of a posteriori strategies for computing error gradients used for directional refinement; and Agouzal et al. [23] proposed a new method for computing tensor metrics provided that an edge-based a posteriori error estimate is given. Moreover, Dobrowolski et al. [24] have pointed out that error estimation based on solving local error problems can be inaccurate on anisotropic meshes. This shortcoming of local error estimates can be explained by the fact that they generally do not contain enough directional information of the solution, which is global in nature, and that their accuracy and effectiveness are sensitive to the aspect ratio of elements, which can be large for anisotropic meshes. We thus choose to develop our approach based on error estimation by means of globally defined error problem. To enhance the computational efficiency, we employ an iterative method to obtain a cost-efficient approximation to the solution of the corresponding global linear system. Numerical results show that a few symmetric Gauß–Seidel iterations are sufficient for this purpose. This is not surprising since the approximation is used only in mesh generation and it is often unnecessary to compute the mesh to a very high accuracy as for the solution of the underlying differential equation. Numerical experiments also show that the new approach is comparable in accuracy and efficiency to methods using Hessian recovery. We also test it with a more challenging example: a heat conduction problem for a thermal battery with large and orthotropic jumps in the material coefficients.[1]

The outline of the paper is as follows. In Section 2, the new framework of using a posteriori hierarchical error estimates for anisotropic mesh adaptation in finite element approximation is described. In Section 3, the optimal metric tensor based on the interpolation error is developed. Several implementation issues are addressed in Section 4. Numerical results obtained with the new approach and with Hessian recovery-based methods are presented in Section 5 for a selection of test examples. Numerical results for the heat conduction problem are given in Section 6. Finally, Section 7 contains conclusions and comments.

2. Model problem and adaptive finite element approximation

In this section, we describe a new framework of using a posteriori hierarchical error estimates for anisotropic mesh adaptation in finite element approximation.

2.1. Model problem and finite element approximation

Consider the boundary value problem of a second-order elliptic differential equation. Assume that the corresponding variational problem is given by

$$(P)\quad \begin{cases} \text{Find } u \in V \text{ such that} \\ a(u, v) = F(v) \quad \forall v \in V, \end{cases}$$

where V is an appropriate Hilbert space of functions over a domain $\Omega \in \mathbb{R}^2$, $a(\cdot,\cdot)$ is a bilinear form defined on $V \times V$, and $F(\cdot)$ is a continuous linear functional on V. The finite element approximation u_h of u is the solution of the corresponding variational problem on a finite dimensional subspace $V_h \subset V$, i.e.,

$$(P_h)\quad \begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h. \end{cases}$$

[1] A Sandia National Laboratories benchmark problem.