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High-order pole-treatment in cylindrical coordinates for incompressible flow simulations with finite-difference collocated schemes

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ABSTRACT

A finite-difference collocated scheme is proposed for the simulation of unsteady incompressible flows in cylindrical coordinates. The Rhie–Chow flux-interpolation technique is used to avoid the odd–even decoupling of the pressure field. It is shown that the calculation of the Coriolis and centrifugal acceleration terms based on the cell-center values lead to numerical instabilities near the pole in the inviscid regime. To avoid this issue, a new formulation based on the combination of cell-center and cell-face velocity components is proposed for the calculation of these terms. Furthermore, a new pole treatment methodology is introduced to calculate the viscous terms, resulting single-valued properties at the pole. As regards the extensions of the present scheme to higher orders of accuracy, a fourth-order spatial discretization is also introduced and verified. Moreover, it is shown that for higher-order extensions, the present pole treatment is able to preserve the order of accuracy and the L_2-norm and L_∞-norm errors are greatly improved in comparison with the previous schemes. The robustness of the present method is assessed via five different numerical tests including laminar flows in a polar cavity, vortex-wall interaction, evolution of a monopole vortex-system to a tripole vortex-system, advection of the lamb dipole, and large eddy simulation (LES) of turbulent pipe flows. Compared to the previous LES results for turbulent pipe flows, significant improvement is obtained for the mean and fluctuating velocities especially at the core region of the channel. Similarly, at the wall region, the fluctuating axial and azimuthal velocities and the skewnesses of the axial and radial velocities are very close to the previous direct numerical simulation and experimental data.

1. Introduction

Finite-difference schemes in cylindrical coordinates are among the most popular methods in numerical simulation of flows in the cylindrical geometry. This is mainly because of the simplicity of discretization on the structured-grid system, orthogonality of principal directions, and homogeneity along the azimuthal and axial directions that facilitate the averaging and filtering operations for large eddy simulation (LES) or direct numerical simulation (DNS). On the other hand, due to the

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singularity of the governing equations at the pole, the structured-grid system introduces numerical difficulties. Accordingly, the search for a robust numerical scheme in cylindrical coordinates has been the subject of many studies thus far [1–10].

The reliability of any finite-difference scheme in the cylindrical coordinates (on either staggered [3–5,8–11], or collocated [1,2,7] grids) greatly depends on the method with which the singularity is treated at the pole. The existing pole-treatment methods can be categorized as

- Methods to remove the singularity, including: i) locally superposing the Cartesian coordinates on the pole [7], ii) applying l'Hopital's rule to the singular terms [1,4–6], iii) interpolating the parameters at the pole [1,5,6,8–10], and iv) deriving segregated sets of governing equations at the pole [4–6].
- Methods that avoid a boundary condition at the pole, including: i) one-sided differentiation on the nodes adjacent to the pole [3,6], and ii) transformation of the governing equations over the whole computational domain [2,3].

Each of these methods has its own benefits and disadvantages. For example, in spite of its simplicity, superposing the Cartesian coordinates on the pole induces strong oscillations in the flow [1,7]. Likewise, applying l'Hopital's rule and one-sided differentiation along the radial direction produces smooth solutions [6] but with a possible deterioration of the accuracy in the results [1].

First, let us consider the staggered schemes in which the velocity components are defined on the cell faces and pressure is defined on the cell centers. For the cells adjacent to the pole in this arrangement, at least one of the velocity components (e.g. radial velocity component) or the pressure must be positioned at the pole with radius $r = 0$ [3–5,8–11].

Consequently, if the radial velocity component u_r is located at the pole (as a common choice), then the simplest pole treatment would be to calculate this parameter by interpolation [5,8–10]. Depending on the interpolation technique, the resulting vector might be single-valued or multi-valued. The main issue with this method is that the governing equations are not fully satisfied at a discrete level. A method to avoid this problem is to apply the spatial derivatives to the ru_r term (rather than u_r), as this term clearly goes to zero at the pole. This method was first introduced by Verzicco and Orlandi [3] in the form of a new set of governing equations for finite difference. The method is second-order accurate in space but the accuracy of this method is compromised on coarse grids, which can be attributed to the one-sided treatment of viscous terms in the cells adjacent to the pole. Moreover, the extension to higher orders is not straightforward in this method.

In another approach, Morinishi et al. [4] derived a non-singular momentum equation at the pole for the evaluation of the radial velocity component, u_r. Their method is also second-order accurate in the radial direction, although it can deliver higher orders of accuracy in the azimuthal and axial directions. The limitation of this method, as pointed out by Desjardins et al. [5], is the unphysical disturbances that might rise near the pole. In order to resolve this issue, Desjardins et al. [5] proposed an alternative equation for u_r using the Taylor expansions. Nevertheless, they preferred to reconstruct u_r at the pole by second-order interpolation to ensure convergence. Although the method is literally high order in all directions, the second-order interpolation at the pole might deteriorate higher orders of accuracy especially on coarse grids or in flows with large gradients at the pole.

As regards the collocated schemes, in which the pressure and velocity components are defined on the cell centers, two main choices are available for the pole treatment. The first one is to position the first cell-center on the pole [1], and the second choice, which is very similar to the staggered schemes, is to shift the first cell-center half a cell-width ($\delta r/2$) away from the pole [2].

The first choice leads to singular terms in the governing equations. To avoid solving such singular equations in compressible flows, Constantinescu and Lele [1] suggested interpolating all the parameters on the pole by means of high-order series expansions. For incompressible flows, however, a pressure Poisson equation has to be solved to decouple the pressure from the momentum equations. In this respect, the mass conservation depends on the consistency between the discrete Poisson equation and the discrete convective terms of the momentum equations [12]. This consistency can be deteriorated if the velocity components are interpolated at the pole and might lead to divergence of the calculation [4].

This problem can be treated by using the second choice explained above; by positioning the first cell-center half a cell-width away from the pole (i.e. avoiding the singular terms). In this case, proper boundary conditions are needed to be applied at the pole. For example, Mohseni and Colonius [2] proposed to map the polar domain from $[0, r] \times [0, 2\pi)$ to $[-r, r] \times [0, \pi)$ for compressible-flow simulations. Inevitably, one of the difficulties would be to solve two different sets of equations for the two ranges of $[0, \pi)$ and $[\pi, 2\pi)$.

In spite of these difficulties, the programming of collocated schemes and particularly their pole treatments in cylindrical coordinates, are generally less problematic compared to the staggered schemes. Also, the available high-order staggered schemes show deficiencies in providing high orders of accuracy at the pole especially regarding the spatial derivatives in the radial direction. Nonetheless, there are only a few studies in the literature that report on the use of the collocated grid in cylindrical coordinates, and to the best of the authors' knowledge, these studies are all limited to compressible flows [1,2].

The other difference between the staggered and collocated grid configurations is their kinetic energy conservation properties. Morinishi et al. [12] introduced discrete conservative forms of momentum and kinetic energy equations on Cartesian and uniform mesh for staggered grid configuration. In their schemes the only source of dissipation of kinetic energy is the time marching algorithm, which introduces a dissipative error proportional to the cube of the computational time step. The extension of their method to cylindrical coordinates (Morinishi et al. [4]) had similar conservative properties for high-order spatial discretization. However as mentioned earlier, the new scheme was still second-order accurate for the spatial