Journal of Computational Physics 296 (2015) 90–100

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

On the convergence of the modified elastic–viscous–plastic method for solving the sea ice momentum equation

Madlen Kimmritz [a,*], Sergey Danilov [a,b], Martin Losch [a]

[a] Alfred-Wegener-Institute for Polar and Marine Research, Bremerhaven, Germany
[b] A.M. Obukhov Institute of Atmospheric Physics RAS, Moscow, Russia

ARTICLE INFO

Article history:
Received 15 January 2015
Received in revised form 28 April 2015
Accepted 29 April 2015
Available online 6 May 2015

Keywords:
Viscous–plastic rheology
Elastic–viscous–plastic rheology
Sea ice
FESOM

ABSTRACT

Most dynamic sea ice models for climate type simulations are based on the viscous–plastic (VP) rheology. The resulting stiff system of partial differential equations for ice velocity is either solved implicitly at great computational cost, or explicitly with added pseudo-elasticity (elastic–viscous–plastic, EVP). A recent modification of the EVP approach seeks to improve the convergence of the EVP method by re-interpreting it as a pseudotime VP solver. The question of convergence of this modified EVP method is revisited here and it is shown that convergence is reached provided the stability requirements are satisfied and the number of pseudotime iterations is sufficiently high. Only in this limit, the VP and the modified EVP solvers converge to the same solution. Related questions of the impact of mesh resolution and incomplete convergence are also addressed.

1. Introduction

The basis of most current sea-ice models is the assumption of viscous–plastic (VP) rheology connecting the ice deformation rates with stresses in the ice [1]. The resulting set of equations is very stiff due to the non-linearity in the VP rheology. Hence, they are computationally challenging and require efficient solution methods to avoid the restriction to very small time steps in standard explicit methods. Partial linearisation allows the stiff part of the problem to be treated implicitly [2]; this requires using solvers but lifts the time step limitation. However, because of linearisation, that is, splitting the operator into implicit and explicit parts and estimating viscosity using the previous Picard iterate, far too many (Picard) iterations ($O(10^4)$) are required to achieve method convergence, so that traditionally only a few iterations are made and convergence is sacrificed [3]. This motivated the implementation of fully nonlinear Jacobian-free Newton–Krylov (JFNK) solvers [4–6]. They converge faster but still remain an expensive solution.

The elastic–viscous–plastic (EVP) method is an alternative to implicit methods. It relaxes the time step limitation of the explicit VP method by introducing an additional elastic term to the stress equations. This allows a fully explicit implementation with much larger time steps than for the explicit VP method [7,8] but requires subcycling within the external time step set by the ocean model. The effects of the additional elasticity term, however, are reported to lead to noticeable differences in the deformation field, and to lead to smaller viscosities and weaker ice (e.g., [5,9–11]). In many cases these effects are linked to the violation of stability limits (analogous to the CFL-criterion for advection) associated with the explicit time stepping scheme of the subcycling process [7,8]. Their most frequent manifestation is grid-scale noise in the ice divergence

* Corresponding author.
E-mail addresses: madlen.kimmritz@awi.de (M. Kimmritz), sergey.danilov@awi.de (S. Danilov), martin.losch@awi.de (M. Losch).

http://dx.doi.org/10.1016/j.jcp.2015.04.051
0021-9991/

field and hence in viscosities. While the numerical code as a rule remains stable and maintains smooth distributions of ice concentration and thickness, the noise in the ice velocity divergence may deteriorate solutions, in particular on meshes with fine or variable resolution [10]. In an effort to improve the performance of the EVP method, a modification of the time-discrete EVP model (EVP*) was proposed by adding an inertial time stepping term to the momentum balance [5]. This EVP* method was further reformulated by [11] as a "pseudotime" iterative scheme converging to the VP rheology. By construction, it should lead to solutions identical to those of the VP method provided it converges and remains stable. Yet, despite improvements in solutions, the convergence has not been achieved [11].

Here we reconsider the elementary analysis of stability of the EVP* method carried out by [11] and conduct a series of numerical simulations that are aimed at clarifying conditions under which the convergence can be achieved. This is the main question that we address in this paper. Additionally, we are going to illustrate the implications of our findings as the resolution is refined. We also explore the consequences of incomplete convergence (limited by the prescribed number of pseudotime steps) on the quality of the EVP* solution.

We start with an introduction of the EVP* scheme as formulated in [11] and elaborate on the convergence conditions of a simplified one-dimensional (1D) scheme. Although this analysis largely follows that by [11], we arrive at new conclusions that help to formulate an optimal strategy. Subsequently, we discuss our results on the basis of experiments performed with the unstructured-mesh finite-element sea ice model FESIM [12], which is a component of the Finite-Element Sea ice–Ocean Model FESOM [13]. Finally, conclusions and outlook are presented.

2. The EVP* method

The horizontal momentum balance of sea ice is written as

$$m(\partial_t + \mathbf{f}\times)\mathbf{u} = a\boldsymbol{\tau} - C_d a\rho_o(\mathbf{u}-\mathbf{u}_o)|\mathbf{u}-\mathbf{u}_o| + \mathbf{F} - mg\nabla H. \tag{1}$$

Here m is the ice (plus snow) mass per unit area, $\mathbf{f}$ is the Coriolis vector, a the ice compactness, $\mathbf{u}$ and $\mathbf{u}_o$ the ice and ocean velocities, ρ_o is the ocean water density, $\boldsymbol{\tau}$ the wind stress, H the sea surface elevation, g the acceleration due to gravity and $F_j = \partial\sigma_{ij}/\partial x_i$ the contribution from stresses within the ice. We follow [11] in writing the VP constitutive law as

$$\sigma_{ij}(\mathbf{u}) = \frac{P}{2(\Delta+\Delta_{\min})}[(\dot\epsilon_{kk}-\Delta)\delta_{ij} + \frac{1}{e^2}(2\dot\epsilon_{ij} - \dot\epsilon_{kk}\delta_{ij})], \tag{2}$$

where

$$\dot\epsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i), \quad \text{and} \quad \Delta = \left(\dot\epsilon_d^2 + \frac{1}{e^2}\dot\epsilon_s^2\right)^{1/2}.$$

The parameter $e = 2$ is the ratio of the major axes of the elliptic yield curve, $\dot\epsilon_d = \dot\epsilon_{kk}$ is the divergence, and $\dot\epsilon_s = ((\dot\epsilon_{11}-\dot\epsilon_{22})^2 + 4\dot\epsilon_{12}^2)^{1/2}$ is the shear. Note that we use the replacement pressure, $(\Delta/(\Delta+\Delta_{\min}))P$, [14] in the formulation of the VP constitutive law to ensure that the stress is on elliptic yield curve when $\Delta \lesssim \Delta_{\min}$. The ice strength P is parameterised as $P = hP^* e^{-c(1-a)}$, where h is the mean thickness, and the constants P^* and c are set to $P^* = 27\,500\ \mathrm{N\,m^{-2}}$ and $c = 20$.

As mentioned above, the difficulty in the integration of (1) is the stiff character of the stress term, which requires prohibitively small time steps in an explicit time stepping scheme. The traditional approach is either implicit [2], where viscosities are estimated at the previous iteration and several iterations are made, or EVP [7,15], which reduces the time step limitations by adding pseudo-elasticity. Discussion of the convergence issues can be found, for example, in [11] and is not repeated here.

The suggestion by [11] is equivalent, up to detail of treating the Coriolis and ice–ocean drag terms, to formulating the EVP* method as:

$$\sigma_{ij}^{p+1} = \sigma_{ij}^p + \frac{1}{\alpha}\left(\sigma_{ij}(\mathbf{u}^p) - \sigma_{ij}^p\right), \tag{3}$$

$$\mathbf{u}^{p+1} = \mathbf{u}^p + \frac{1}{\beta}\left(\frac{\Delta t}{m}\nabla\cdot\boldsymbol{\sigma}^{p+1} + \frac{\Delta t}{m}\mathbf{R}^{p+1/2} + \mathbf{u}_n - \mathbf{u}^p\right). \tag{4}$$

In (4), $\mathbf{R}$ sums all the terms in the momentum equation except for the rheology and the time derivative, Δt is the time step of the ice model, the index n labels the time levels, that is, discrete moments in the real time, and the index p is that of pseudotime (subcycling step number). The Coriolis term in $\mathbf{R}^{p+1/2}$ is treated implicitly in our implementation and the ice–ocean stress term is linearly-implicit ($C_d\rho_o|\mathbf{u}_o - \mathbf{u}^p|(\mathbf{u}_o - \mathbf{u}^{p+1})$). In (3), $\sigma_{ij}(\mathbf{u}^p)$ implies that the stresses are estimated by (2) based on the velocity from iteration p, and σ_{ij}^p is the variable of the pseudotime iteration. The parameters α and β in the last formulae are large numbers that are selected from stability considerations [11]. They replace the terms $2T/\Delta t_e$ and $(\beta^*/m)(\Delta t/\Delta t_e)$, where T is the elastic damping time scale and Δt_e the subcycling time step of the standard EVP formulation, and parameter β^* has been introduced in [5]. After convergence of (3) and (4), the pseudotime terms drop out and the resulting solution is exactly the VP solution:

$$\frac{m}{\Delta t}\left(\mathbf{u}_{n+1} - \mathbf{u}_n\right) = \nabla\cdot\boldsymbol{\sigma}(\mathbf{u}_{n+1}) + \mathbf{R}^*, \tag{5}$$