Journal of Computational Physics 296 (2015) 369–381

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Stability of the discretization of the electron avalanche phenomenon

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ARTICLE INFO

Article history:
Received 29 September 2014
Received in revised form 17 March 2015
Accepted 11 May 2015
Available online 14 May 2015

Keywords:
Advection
Reaction
Streamer

ABSTRACT

The numerical simulation of the discharge inception is an active field of applied physics with many industrial applications. In this work we focus on the drift-reaction equation that describes the electron avalanche. This phenomenon is one of the basic building blocks of the streamer model. The main difficulty of the electron avalanche equation lies in the fact that the reaction term is positive when a high electric field is applied. It leads to exponentially growing solutions and this has a major impact on the behavior of numerical schemes. We analyze the stability of a reference finite volume scheme applied to this latter problem. The stability of the method may impose a strict mesh spacing, therefore a proper stabilized scheme, which is stable whatever spacing is used, has been developed. The convergence of the scheme is treated as well as some numerical experiments.

1. Introduction

In this paper we analyze the stability of a reference finite volume scheme applied to the solution of a three-dimensional linear advection–reaction equation with a strongly positive reaction term. This problem is representative of a kind of instability affecting the discretization of the electron avalanche: one of the main physical mechamisms of the streamer problem when a high electric field is considered [1,2].

The streamer model gathered a lot of attention in the past years since it had many applications to the ageing of electrical components [3], the surface treatment of polymers [4], the pollution reduction [5] and the production of chemically active species [6].

The numerical simulation of this model became popular as well. This is a complex interplay among the transport of charged species, the electric field and the chemical reactions between charged and neutral species. In particular the solution of the hyperbolic drift problem, representing the advection of charged particles, got most of the attention. One of the first approaches used was the finite difference method [7]. The flux corrected method also became very popular [8] and in more recent years also the finite element [9,10] and finite volume methods were applied [11–13]. In [14] an analysis of the performances of the advection stabilization techniques applied to the streamer problem has been carried out. Also the discontinuous Galerkin method was tested in [15].

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http://dx.doi.org/10.1016/j.jcp.2015.05.013
0021-9991/

Although there is a large amount of literature concerning the simulation of this particular problem there are few references treating the stability of these methods. An analysis on the interaction between the charged species movement and the electric field and their implications on the stability of the discretizations of the streamer model was carried out in [16–18].

With respect to those works we want to study the interactions between the transport of charged species and the chemistry. To the best of the authors' knowledge this aspect has never been treated in detail in a three dimensional geometry. In particular we want to treat the stability of a linear advection–reaction equation that represents the electron avalanche. In this case there is a net production of electrons and this was demonstrated to be a possible cause of instabilities, see [19].

Although there is a large literature covering the stability of the discretization of diffusion–advection–reaction problems [20–22] and a less abundant one covering the advection–reaction problems [23], there is no explicit reference to unstable reaction terms: i.e. only sink terms are usually considered.

As already outlined in [19], the stability of the transport and reaction problem is strongly dependent on the mesh spacing. In particular the mesh should be kept very fine where the electric field is strong and this can lead to unacceptable high computational costs. To this end, and to maximize the accuracy, many local refinement techniques were developed and applied to the streamer problem [24–26].

In this work we propose a complementary approach and we derive a method that is stable whatever mesh spacing is used. Let us now review the structure of this paper: in Section 2 we introduce the continuous reference problem and we study its properties. These latter ones depend strongly on the structure of the advection field. In 3 we introduce the discrete method and in 4 we study the implications of the structure of the advection field on the discrete method. The analysis of the properties of our scheme is included in 5 and some numerical experiments are described in 6.

2. The continuous problem

Let $\Omega \subset \mathbb{R}^3$ be an open domain with Lipschitz boundary $\partial\Omega$, let $\vec{n}(\vec{x})$, with $\vec{x} \in \partial\Omega$, be the outward pointing unit vector and, finally, let $(0, t^{fin}]$ be a time interval where t^{fin} is the final time, not necessarily bounded. In this context Ω represents a portion of air where the charged particles move. We consider a linear advection equation with a positive reaction term:

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \left(u\vec{c}\right) = \alpha \left|\vec{c}\right| u, \tag{1}$$

where u is the solution (in the streamer equation [1] it represents the concentration of the electrons), $t \in (0, t^{fin}]$ is the time variable, $\vec{c}(\vec{x})$ is a given Lipschitz-continuous velocity field (with Lipschitz constant θ_L), $\vec{x} \in \Omega$ is the position vector and $\alpha(\vec{x}) > 0$ is a bounded, strictly positive, reaction coefficient. Eq. (1) represents the dominant part of the electron evolution equation, see for instance [27,28], when a very high electric field is considered. The velocity field is computed using

$$\vec{c} = -\mu\vec{E} = \mu\vec{\nabla}\phi, \tag{2}$$

where $\mu > 0$ is the electron mobility that we have considered constant for simplicity. Moreover $\vec{E}$ is the electric field and ϕ is the electric potential. We also require that the electric field satisfies the electrostatic equation

$$\vec{\nabla} \cdot \vec{E} = \frac{q}{\varepsilon}, \tag{3}$$

where $q(\vec{x})$ is a given, bounded, space charge density and ε is the permittivity.

Eq. (1) must be complemented by a set of initial and boundary conditions in particular we set $u(0, \vec{x}) = u_0(\vec{x})$ with $\vec{x} \in \Omega$ where $u_0(\vec{x}) \geq 0$ is a positive bounded function. Moreover let $\partial\Omega_{in} = \{\vec{x} \in \partial\Omega : \vec{c}(\vec{x}) \cdot \vec{n} < 0\}$ be the inflow boundary, then

$$u(t, \vec{x}) = u_b(\vec{x}),\ \ \vec{x} \in \partial\Omega_{in}, \tag{4}$$

where $u_b \geq 0$ is a bounded positive function. Finally let $\partial\Omega_{out} = \{\vec{x} \in \partial\Omega : \vec{c}(\vec{x}) \cdot \vec{n} \geq 0\}$ be the outflow part of the boundary.

We can rewrite Eq. (1) using the technique of the characteristics [29]. Let

$$\begin{cases} \frac{d\vec{X}(t)}{dt} = \vec{c}(\vec{X}(t)), \\ \vec{X}(0) = \vec{X}_0, \end{cases} \tag{5}$$

be the ordinary differential equation (ODE) for the evolution of the characteristics where $\vec{X}(t)$ is their trajectory and the point $\vec{X}_0 \in \Omega$ is their root. The solution on the characteristics can be computed using

$$\begin{cases} \frac{du(t,\vec{X}(t))}{dt} = \left(\alpha(\vec{X}(t)) \left|\vec{c}(\vec{X}(t))\right| - \vec{\nabla} \cdot \vec{c}(\vec{X}(t))\right) u(t, \vec{X}(t)) = \left(\alpha(\vec{X}(t)) \left|\vec{c}(\vec{X}(t))\right| - \mu \frac{q(\vec{X}(t))}{\varepsilon}\right) u(t, \vec{X}(t)), \\ u(0, \vec{X}_0) = u_0(\vec{X}_0), \end{cases} \tag{6}$$

where we have used (2) and (3) to simplify the right hand side. The existence of the solution of the first ODE is guaranteed by the fact that the field vector $\vec{c}$ is Lipschitz. For the second ODE (6) we observe that the forcing term is the sum of two parts. The first one, i.e. $\alpha|\vec{c}|u$, is the product of a bounded function, a Lipschitz-continuous function, and the solution. Therefore the first term is Lipschitz with respect to u. A similar argument holds for the second term $\mu\frac{q(\vec{X}(t))}{\varepsilon}u$ and, thus, a unique solution exists.